XIV Encuentro de Invierno Geometría, Mecánica y Teoría de Control
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Some problems in Geometric Mechanics

Geometric Mechanics, Classical field theories and Poisson structures

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To study, from a geometric perspective, different methods to simplify how obtain

- the trajectories of a mechanical system (subject or not to non-holonomic constraints)
- solutions of dynamic equations in first order classical field theories

Several geometric methods

- Reduction Theory
- Hamilton-Jacobi Theory
- Invariant volumes
- Geometric integrators

Geometric tools

- Poisson structures and generalizations
- Multisymplectic structures

- **Reduction Theory**
  - Reduction of symplectic Lie algebroids
  - Reduction of the Jacobi structure on the sphere of the dual bundle to a Lie algebroid endowed with a bundle metric
  - Reduction and reconstruction of non-autonomous hamiltonian systems and symplectic principal $\mathbb{R}$-bundles
  - Introduction of the notion of a multi-Poisson structure as the Poisson version of a multisymplectic structure. Application to the reduction of first-order classical field theories
  - Marsden-Weinstein multisymplectic reduction. Application to the reduction of first-order classical field theories
- **Hamilton-Jacobi Theory**
  - Hamilton-Jacobi theory for hamiltonian systems with respect to linear Poisson structures
  - Hamilton-Jacobi theory and complete integrability for (generalized) non-holonomic mechanical systems
  - Hamilton-Jacobi theory for first-order classical field theories in the Lie algebroid setting
- **Tulczyjew's triple for first-order classical field theories in the Lie algebroid setting**
- **Study of invariant volume forms for non-holonomic mechanical systems**
- **Introduction of the intrinsic notion of Poisson symmetric space**
- **Local description on discrete mechanics on Lie groupoids. The exact discrete lagrangian on a Lie groupoid**

Autonomous hamiltonian systems

- The configuration space: Q
- The phase space of momenta: T^*Q
- Hamiltonian function: $H : T^*Q \to \mathbb{R}$
- Structure: The canonical symplectic structure $\Omega_Q \in \Omega^2(T^*Q)$
- Hamiltonian vector field: $\mathcal{H}_H^{\Omega_Q} \in \mathfrak{X}(T^*Q)$, $i_{\mathcal{H}_H^{\Omega_Q}} \Omega_Q = dH$
- Solutions of Hamilton equations: integral curves of $\mathcal{H}_H^{\Omega_Q}$

In the presence of a group of symmetries

$\Downarrow$

The system can be reduced to a system with less freedom degrees

Reduction and reconstruction processes $\Rightarrow$ the solutions of Hamilton equations

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}, \qquad \frac{dq}{dt} = \frac{\partial H}{\partial p}$$

Marsden-Weinstein reduction Theorem

INGREDIENTS:

- (M, Ω) symplectic manifold
- $\phi : G \times M \to M$ a free and proper symplectic action of a Lie group G
- $J : M \to \mathfrak{g}^*$ an Ad^*-equivariant momentum map associated with ϕ

$$\xi_M = \mathcal{H}^{\Omega}_{J_\xi}, \quad J(\phi_g(x)) = \mathrm{Ad}^*_{g^{-1}}(J(x)), \qquad \text{for any } \xi \in \mathfrak{g}, x \in M$$

- $\nu \in \mathfrak{g}^*$

$$\phi_\nu : G_\nu \times J^{-1}(\nu) \to J^{-1}(\nu), \quad G_\nu = \{g \in G / \mathrm{Ad}^* \nu = \nu\}$$

$$\Downarrow$$

$(J^{-1}(\nu)/G_\nu, \Omega_\nu)$ is a symplectic manifold

$$\pi_\nu^* \Omega_\nu = i_\nu^* \Omega$$

$$\pi_\nu : J^{-1}(\nu) \to J^{-1}(\nu)/G_\nu, \quad i_\nu : J^{-1}(\nu) \to M$$

Marsden-Weinstein reduction Theorem by stages

$$G = H_1 \times H_2$$

Reduce by the subgroup H_1 at a time and then to reduce by H_2

$$\Updownarrow$$

Reduce by G

Some Applications

- underwater vehicle dynamics and stability (Leonard and Marsden [1997])
- Applications to compressible fluids (Holm and Kupershmidt [1983])

H closed and normal subgroup of G

Reduce by H and then by a group related with G/H

$$\Updownarrow$$

Reduce by G

- COTANGENT REDUCTION
- KIRILLOV-KOSTANT-SOURIAU THEOREM

- $(M, \Omega) = (T^*Q, \Omega_Q)$
- $\phi : G \times Q \to Q$ a free and proper action
    - $T^*\phi : G \times T^*Q \to T^*Q$ free and proper action
    - $J : T^*Q \to \mathfrak{g}^*$ Ad^*-equivariant hamiltonian momentum map

$$J(\alpha_q)(\xi) = \alpha_q(\xi_Q(q))$$

- $\nu \in \mathfrak{g}^*$

$$\Downarrow$$

$(J^{-1}(\nu)/G_\nu, (\Omega_Q)_\nu)$ reduced symplectic manifold

$$+$$

$\alpha_\nu \in \Omega^1(Q)$ G_ν-invariant 1-form with values in $J^{-1}(\nu)$

$$\Downarrow$$

$\varphi_{\alpha_\nu} : (J^{-1}(\nu)/G_\nu, (\Omega_Q)_\nu) \to (T^*(Q/G_\nu), \Omega_{Q/G_\nu} - B_{\alpha_\nu})$ embedding

φ_{α_ν} is a symplectomorphism $\iff \mathfrak{g} = \mathfrak{g}_\nu$

Kirillov-Kostant-Souriau Theorem

The coadjoint orbits in the dual space of the Lie algebra of a Lie group admit a symplectic structure

INGREDIENTS

- G a Lie group with Lie algebra $\mathfrak{g}$
- The free and proper action $l : G \times G \to G, \qquad l(g,h) = l_g(h) = gh.$
- The symplectic manifold $(T^*G \cong G \times \mathfrak{g}^*, \Omega_G)$
- The symplectic action $T^*l : G \times G \times \mathfrak{g}^* \to G \times \mathfrak{g}^*$
- The Ad^*-equivariant momentum map $J^{T^*G} : G \times \mathfrak{g}^* \to \mathfrak{g}^*$

$$(J^{T^*G})^{-1}(\nu) = \{(g, \mathrm{Ad}^*_g \nu) | g \in G\} \simeq G$$

$\Downarrow$ Marsden-Weinstein symplectic reduction theorem

$$(J^{T^*G})^{-1}(\nu)/G_\nu \cong G/G_\nu \simeq \mathcal{O}_\nu$$

INGREDIENTS

- Poisson manifold $(M, \{\cdot,\cdot\})$
- $\phi : G \times M \to M$ a free and proper canonical Poisson action

$$\Downarrow$$

M/G a Poisson manifold

$$\{f_1, f_2\}_r \circ \pi = \{f_1 \circ \pi, f_2 \circ \pi\},$$

where $\pi : M \to M/G$ is the canonical projection and $f_1, f_2 \in C^\infty(M/G)$

$M = T^*Q \quad T^*\phi : G \times T^*Q \to T^*Q$ with $\phi : G \times M \to M$ free and proper action

$T^*Q/G \to Q/G$ linear Poisson manifold

$Q = G$ and $\phi = I : G \times G \to G$

$$\Downarrow$$

the symplectic leaves of T^*G/G are the coadjoint orbits

To extend the reduction process for linear Poisson structures on vector bundles

- $A \to M$ vector bundle

$\{\Lambda_{A^*} \in \mathcal{V}^2(A^*)$ linear Poisson structure on $A^*\} \leftrightarrow \{(\llbracket \cdot, \cdot \rrbracket, \rho)$ Lie algebroid structure on $A\}$

- $\llbracket \cdot, \cdot \rrbracket : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ Lie bracket
- $\rho : A \to TM$ vector bundle morphism (anchor map)

$$\llbracket X, fY \rrbracket = f \llbracket X, Y \rrbracket + \rho(X)(f) Y, \quad \forall X, Y \in \Gamma(A), \quad \forall f \in C^\infty(M)$$

$$\Downarrow$$

$(A, \llbracket \cdot, \cdot \rrbracket, \rho)$ Lie algebroid

- Schouten bracket $\llbracket \cdot, \cdot \rrbracket : \Gamma(\wedge^p A) \times \Gamma(\wedge^{p'} A) \to \Gamma(\wedge^{p+p'-1} A)$
- The differential of A $d^A : \Gamma(\wedge^k A^*) \to \Gamma(\wedge^k A^*)$

$$(A \to M, [\![\cdot,\cdot]\!], \rho) \text{ Lie algebroid of rank } 2n + \Omega \in \Gamma(\wedge^2 A^*)$$

$$d^A\Omega = 0, \quad (\Omega)^n \neq 0$$

$$\Downarrow$$

M Poisson manifold

EXAMPLE:

$$(\tau : A \to M, [\![\cdot,\cdot]\!], \rho) \text{ Lie algebroid of rank n}$$

$\mathcal{T}^A A^* \to A^*$ vector bundle

$$(\mathcal{T}^A A^*)_{a^*} = \{(a, v) \in A_{\tau^*(a^*)} \times T_{a^*}A^* / \rho(a) = T_{a^*}\tau^*(v)\}$$

$\mathcal{T}^A A^* \to A^*$ is a Lie algebroid of rank $2n$ $\equiv$ the cover of fiberwise linear Poisson manifold

$$\lambda_A \in \Gamma((\mathcal{T}^A A^*)^*), \quad \Omega_A = -d^{\mathcal{T}^A A^*}\lambda_A$$

$$\Downarrow$$

$\mathcal{T}^A A^* \to A^*$ is a symplectic Lie algebroid

- Reduction process for Lie algebroids
- Reduction process for symplectic Lie algebroids in the presence of a momentum map
- Application to the symplectic cover of a fiberwise linear Poisson manifold

J.C. Marrero, E. Padrón, M. Rodríguez-Olmos: Reduction of a symplectic Lie algebroid with momentum map and its application to fiberwise linear poisson structures Preprint 2011.

INGREDIENTS

- $(A \to M, [\![\cdot,\cdot]\!], \rho, \Omega)$ symplectic Lie algebroid
- $\Phi : G \times A \to A$ action by complete lifts associated with $\phi : G \times M \to M$

$\psi : \mathfrak{g} \to \Gamma(A)$ Lie algebra anti-morphism such that $\Phi^* : G \times A^* \to A^*$ action by Poisson automorphisms

$$\xi_{A^*} = \mathcal{H}^{\Omega}_{\widehat{\psi(\xi)}}$$

$$\Downarrow$$

$\Phi^T : TG \times A \to A$ an affine action of $TG \cong G \times \mathfrak{g}$ over A

$$\Phi^T((g,\xi), a_x) = \Phi_g(a_x + \psi(\xi)(x))$$

$$\Downarrow$$

A/TG is a Lie algebroid over M/G and $\widetilde{\pi} : A \to A/TG$ is a Lie algebroid epimorphism

INGREDIENTS

$(A \to M, [\![\cdot,\cdot]\!], \rho, \Omega)$ symplectic Lie algebroid

$\Phi : G \times A \to A$ action by complete lifts

$J : M \to \mathfrak{g}^*$ momentum map

$$Ad^*_{g^{-1}}(J(x)) = J(\phi_g(x)), \quad \forall x \in M, \quad \forall g \in G$$

$$\Phi^*_g(\Omega) = \Omega \text{ and } i_{\psi(\xi)}\Omega = d^A J_\xi, \text{ for all } g \in G \text{ and } \xi \in \mathfrak{g}$$

$$\Downarrow$$

$$J^T : A \to \mathfrak{g}^* \times \mathfrak{g}^*, \quad J^T(a) = ((TJ \circ \rho)(a), J(\tau(a)))$$

$TG \times A \to A$ affine action

$(A_\nu = (J^T)^{-1}(0,\nu)/TG_\nu \to J^{-1}(\nu)/G_\nu, \Omega_\nu)$ symplectic Lie algebroid

INGREDIENTS

- $(\mathcal{T}^A A^* \to A^*, \Omega_A)$ symplectic Lie algebroid
- $\Phi : G \times A \to A$ action by complete lifts with respect to $\psi : \mathfrak{g} \to \Gamma(A)$

$$\Downarrow$$

$$(\Phi, T\Phi^*) : G \times \mathcal{T}^A A^* \to \mathcal{T}^A A^*$$

$$\psi^T : \mathfrak{g} \to \Gamma(\mathcal{T}^A A^*)$$

- $J_{A^*} : A^* \to \mathfrak{g}^*$ hamiltonian momentum map

$$J_{A^*}(\alpha_x)(\xi) = \alpha_x(\psi(\xi)(x))$$

$((\mathcal{T}^A A^*)_\nu, \Omega_\nu) \to (\mathcal{T}^{A_{0,\nu}} A^*_{0,\nu}, \Omega_{A_{0,\nu}} - (pr_1)^* B_\nu)$ symplectic Lie algebroid embedding

isomorphism if and only if $\mathfrak{g} = \mathfrak{g}_\nu$

$$A_{0,\nu} = A/TG_\nu \to M/G_\nu$$

NEW OBJECTIVES:

- To apply this reduction process to new examples
- Bundle version reduction of the symplectic cover of a fiberwise linear Poisson manifold
- Reduction by stages of symplectic Lie algebroids

The cotangent sphere bundle of a Riemannian manifold (Q, g) of dimension n

$$\mathbb{S}^{n-1}(T^*Q) = T_1^*Q = \{\alpha \in T^*Q / g(\alpha, \alpha) = 1\} \to Q$$

$$i_{\mathbb{S}^{n-1}(T^*Q)} : \mathbb{S}^{n-1}(T^*Q) \to T^*Q, \quad \lambda \in \Omega^1(T^*Q) \text{ Liouville 1-form}$$

$$\theta = -i^*_{\mathbb{S}^{n-1}(T^*Q)}(\lambda) \in \Omega^1(\mathbb{S}^{n-1}(T^*Q)) \qquad \theta \wedge (d\theta)^n \text{ is a volume form on } T^*Q$$

Unit sphere of a real Lie algebra with scalar product $(\mathfrak{g}, [\cdot,\cdot], <\cdot,\cdot>)$ of dimension n

$$\mathbb{S}^{n-1}(\mathfrak{g}^*) = \{\alpha \in \mathfrak{g}^* / <\alpha, \alpha> = 1\}$$

$$\Lambda_{\mathfrak{g}^*} \text{ Lie-Poisson structure on } \mathfrak{g}^*$$

$$\bar{\Lambda} = \Lambda_{\mathfrak{g}^*} - \Delta_{\mathfrak{g}^*} \wedge i_{\alpha_{\mathfrak{g}^*}} \Lambda_{\mathfrak{g}^*} \in \mathcal{V}^2(\mathfrak{g}^*), \quad \bar{E} = i_{\alpha_{\mathfrak{g}^*}} \Lambda_{\mathfrak{g}^*} \in \mathfrak{X}(\mathfrak{g}^*),$$

$$(\mathfrak{g}^*, (\bar{\Lambda}, \bar{E})) \text{ Jacobi manifold}$$

$$[\bar{\Lambda}, \bar{\Lambda}] = 2\bar{E} \wedge \bar{\Lambda}, \quad [\bar{E}, \bar{\Lambda}] = 0$$

$$\Downarrow$$

$$(\mathbb{S}^{n-1}(\mathfrak{g}^*), (\bar{\Lambda}_{|\mathbb{S}^{n-1}(\mathfrak{g}^*)}, \bar{E}_{|\mathbb{S}^{n-1}(\mathfrak{g}^*)})) \text{ Jacobi manifold}$$

(A^*, Λ_{A^*}) linear Poisson manifold with $\tau : A \to M$ + g fiber bundle metric on A^*

$$\Downarrow$$

$\mathbb{S}^{n-1}(A^*) \to Q$ Jacobi manifold ?

$$\begin{array}{ccc}
\mathcal{T}^A\mathbb{S}^{n-1}(A^*) & \xrightarrow{\mathcal{T}^A i} & \mathcal{T}^A A^* \\
\downarrow & & \downarrow \\
\mathbb{S}^{n-1}(A^*) & \xrightarrow{i} & A^*
\end{array}$$

$\mathcal{T}^A\mathbb{S}^{n-1}(A^*) \to \mathbb{S}^{n-1}(A^*)$ subalgebroid of $\mathcal{T}^A A^* \to A^*$

$\theta_A = -(\mathcal{T}^A i)^*(\lambda_A)$ contact structure

$$\theta_A \wedge (d^A\theta_A)^n \neq 0$$

$\Omega_A = -d^{\mathcal{T}^A A^*}\lambda_A$ symplectic structure

$$\Downarrow$$

$\mathbb{S}^{n-1}(A^*)$ Jacobi manifold

A^* linear Poisson manifold

$$i^*\mathcal{T}^A A^* = Ti(\mathcal{T}^A\mathbb{S}^1(A^*)) \oplus i^*(< \Delta >), \qquad \Delta \in \Gamma(\mathcal{T}^A A^*)$$

D. Chinea, J.C. Marrero, E. Padrón: The Jacobi structure on the sphere of a linear Poisson manifold on a vector bundle with fiber metric, work in progress

New Objetives

- Give an explicit description of the Jacobi structure on $\mathbb{S}^{n-1}(A^*)$ for known examples of Lie algebroids A
- Give a contact reduction procedure for contact Lie algebroids in the presence of momentum map, apply this result to the particular case of $\mathcal{T}\mathbb{S}^{n-1}(A^*)$
- Under certain regularity conditions, study the manifold $\mathbb{S}^{n-1}(A^*)/E$ Is it again a Poisson manifold?

$\mathbb{R}^{2n}$ symplectic manifold $\Rightarrow \mathbb{S}^{n-1}(\mathbb{R}^{2n})$ contact manifold $\Rightarrow \mathbb{S}^{n-1}(\mathbb{R}^{2n})/E \cong \mathbb{C}P^{n-1}$ symplectic manifold

A^* linear Poisson manifold $\Rightarrow \mathbb{S}^{n-1}(A^*)$ Jacobi manifold $\Rightarrow \mathbb{S}^{n-1}(A^*)/E$ Poisson manifold

Non-autonomous hamiltonian systems

- The configuration space: $\pi : M \to \mathbb{R}$ surjective submersion
- The phase space of momenta:
  - extended version: T^*M
  - restricted version: $V^*\pi \quad V_x\pi = T_x\pi$ for all $x \in M$
- Structure: $\mu_\pi : T^*M \to V^*M$

$\mu_\pi : T^*M \to V^*\pi$ **is a principal $\mathbb{R}$-bundle**

$$\psi_\pi : \mathbb{R} \times T^*M \to T^*M, \ \ \psi_\pi(s, \alpha_x) = \alpha_x + s\pi^*(dt)(x)$$

- ψ_π is symplectic

$\mu : (A, \Omega) \to V$ is a *symplectic principal $\mathbb{R}$-bundle* if Ω is a symplectic structure on A such that the associated principal action $\psi : \mathbb{R} \times A \to A$ is symplectic

$$\mu : (A, \Omega) \to (V \cong A/\mathbb{R}, \Lambda) \text{ Poisson morphism}$$

- $\phi : G \times A \to A$ is a *canonical action* on the symplectic principal $\mathbb{R}$-bundle $\mu : (A, \Omega) \to V$
  - ϕ is a symplectic action
  - $\phi_g \circ \psi_s = \psi_s \circ \phi_g$ for any $g \in G$, $s \in \mathbb{R}$
  - the 1-form $\zeta_\mu = i_{Z_\mu}\Omega$ is basic with respect to ϕ, i.e $\zeta_\mu(\xi_A) = 0$ for any $\xi \in \mathfrak{g}$

$$+$$

- $J : A \to \mathfrak{g}^*$ *Ad^*-equivariant momentum map for ϕ*

$$\Downarrow$$

- $\phi^V : G \times V \to V$ Poisson action

$$\phi^V(g, v) = \mu(\phi_g(a))\text{for any } g \in G,\ v \in V$$

- $J^V : V \to \mathfrak{g}^*$ Ad*-equivariant momentum map for ϕ^v

$$J^V(v) = J(a),\ \text{with } a \in \mu^{-1}(v)$$

INGREDIENTS

- $\mu : (A, \Omega) \to V$ symplectic $\mathbb{R}$-principal bundle
- $\phi : G \times A \to A$ canonical action with $\phi^V : G \times V \to V$ is free and proper
- $J : A \to \mathfrak{g}^*$ Ad^*-equivariant momentum map
- $\nu \in \mathfrak{g}^*$

$\Downarrow$ Marsden-Weinstein Th.

$$(A_\nu = J^{-1}(\nu)/G_\nu, \Omega_\nu)$$

$\Downarrow$ Poisson reduction Th.

$$V_\nu = (J^V)^{-1}(\nu)/G_\nu$$

$$\Downarrow$$

$\mu_\nu : (A_\nu, \Omega_\nu) \to V_\nu$ is a symplectic principal $\mathbb{R}$-bundle

$\{\cdot, \cdot\}_\nu$ Poisson on $V_\nu \equiv$ the Poisson bracket induced by $\mu_\nu : A_\nu \to V_\nu$

- $\phi : G \times M \to M$ free and proper action
- $\pi : M \to \mathbb{R}$ a G-invariant surjective submersion
- $\nu \in \mathfrak{g}^*$
- $\lambda_\nu \in \Omega^1(M)$ a G_ν-invariant 1-form with values in $J_\nu^{-1}(\nu')$

Then $\tilde{\pi}_\nu : M/G_\nu \to \mathbb{R}$ is a surjective submersion and there is a symplectic principal $\mathbb{R}$-bundle embedding

$$\begin{array}{ccc} ((T^*M)_\nu, (\Omega_M)_\nu) & \xrightarrow{\varphi_{\lambda_\nu}} & (T^*(M/G_\nu), \Omega_{M/G_\nu} - B_{\lambda_\nu}) \\ \downarrow (\mu_\pi)_\nu & & \downarrow \mu_{\tilde{\pi}_\nu} \\ (V^*\pi)_\nu & \xrightarrow{\varphi^V_{\lambda_\nu}} & V^*\tilde{\pi}_\nu \end{array}$$

$B_{\lambda_\nu} \in \Omega^2(T^*(M/G_\nu))$ magnetic term associated with λ_ν

φ_{λ_ν} is a symplectic principal $\mathbb{R}$-bundle isomorphism $\Leftrightarrow \mathfrak{g} = \mathfrak{g}_\nu$

Kirillov-Kostant-Souriau theorem (version for principal G-bundle with base space $\mathbb{R}$

INGREDIENTS

- $\phi : G \times M \to M$ free and proper action with principal G-bundle projection $\pi : M \to M/G \simeq \mathbb{R}$

$$\Downarrow$$

$$T^*\phi : G \times T^*M \to T^*M \ + \ \text{Ad}^*\text{-equivariant momentum map } J : T^*M \to \mathfrak{g}^*$$

$$\Downarrow$$

$$\mu_\pi : T^*M \to V^*\pi \text{ be the standard symplectic } \mathbb{R}\text{-bundle}$$

- $\nu \in \mathfrak{g}^*$

$$\Downarrow \ \text{symplectic principal } \mathbb{R}\text{-bundle reduction theorem}$$

$$(\mu_\pi)_\nu : ((T^*M)_\nu, (\Omega_M)_\nu) \to (V^*\pi)_\nu \cong M/G_\nu$$

The Poisson structure on M/G_ν

The space of orbits M/G_ν of the action of G_ν on M admits a Poisson structure $\{\cdot,\cdot\}_{M/G_\nu}$ and the symplectic leaf of M/G_ν passing through the point $[x]$ is symplectomorphic to $\mathcal{O}_\nu$

I. Lacirasella, J. C. Marrero, E. Padrón: Reduction of symplectic R-bundles, Preprint 2012, arXiv:1201.4690

- To develop the bundle version of the reduction of the standard symplectic principal $\mathbb{R}$-bundle
- To reconstruct the dynamics of a non-autonomous hamiltonian systems on a symplectic principal $\mathbb{R}$-bundle from reduced dynamics.
- To study the reduction theory by stages of a symplectic principal $\mathbb{R}$-bundle

Classical field theories $\Leftrightarrow$ multisymplectic formulation

$$\text{multisymplectic structure } \Omega \in \Omega^k(M), \quad d\Omega = 0$$

$$i_X\Omega = 0 \;\Rightarrow\; X = 0, \quad \forall X \in \mathfrak{X}(M)$$

EXAMPLE

$M = \wedge^k T^*Q$

$$\lambda_M(\alpha)(X_1, \ldots, X_k) = \alpha(T\pi(X_1), \ldots, T\pi(X_k)),$$

$\pi\colon \wedge^k T^*Q \to Q$.

$\Omega_M = -d\lambda_M$ es una forma multisimpléctica en Q

k-polisymplectic structure on M

$$\Updownarrow$$

$(\omega^1, \dots, \omega^k)$ closed non-degenerate 2-forma $\mathbb{R}^k$-valued

$$\bar{\omega} = \sum_{A=1}^{k} \omega^A \otimes e_A$$

$\{e_1, \dots, e_k\}$ canonical base of $\mathbb{R}^k$.

EXAMPLE 1:
$(T_k^1)^* Q := T^*Q \oplus \stackrel{(k}{\dots} \oplus T^*Q$,

$$\omega^A = (\pi_Q^{k,A})^* \omega,$$

ω canonical symplectic form on T^*Q and $\pi_Q^{k,A} : T^*Q \oplus \stackrel{(k}{\dots} \oplus T^*Q \to T^*Q$ the projection

$((T_k^1)^* Q, \omega^1, \dots, \omega^k)$ polisympletic manifold

G a Lie group and $\mathfrak{g}$ its Lie algebra

$$\mathfrak{g}^* \times \overset{(k}{\ldots} \times \mathfrak{g}^*$$

$$\begin{array}{rcl} \mathrm{Coad}^k \colon G \times \mathfrak{g}^* \times \overset{(k}{\ldots} \times \mathfrak{g}^* & \to & \mathfrak{g}^* \times \overset{(k}{\ldots} \times \mathfrak{g}^* \\ (g, \nu_1, \ldots, \nu_k) & \mapsto & (\mathrm{Coad}\,(g, \nu_1), \ldots, \mathrm{Coad}\,(g, \nu_k)) \end{array}$$

$(\nu_1, \ldots, \nu_k) \in \mathfrak{g}^* \times \overset{(k}{\ldots} \times \mathfrak{g}^*$ *coadjoint k-orbits*

$$\mathcal{O}_{(\nu_1,\ldots,\nu_k)} = \{\mathrm{Coad}^k(g, \nu_1, \ldots, \nu_k) \mid g \in G\},$$

is a polisymplectic manifold

$$\omega^A_\nu \colon = (pr_A)^* \omega_{\nu_A},$$

$pr_A \colon : \mathcal{O}_{(\nu_1,\ldots,\nu_k)} \to \mathcal{O}_{\nu_A}$, $(\nu_1, \ldots, \nu_k), \mapsto \nu_A$ canonical projection and ω_{ν_A} sympletic form of $\mathcal{O}_{\nu_A}$

Ω symplectic structure

$$\Updownarrow$$

$$\flat_\omega : TM \to T^*M, \quad \flat_\omega(v_x) = i_{v_x}\omega_x \text{ vector bundle isomorphism}$$

$$\flat_\omega([X,Y]) = \mathcal{L}_X \flat_\omega(Y) - \mathcal{L}_Y \flat_\omega(X) - \mathrm{d}(\flat_\omega(X)(Y)), \quad \forall X, Y \in \mathfrak{X}(M)$$

(M, Λ) Poisson manifold

$$\Downarrow$$

$$\sharp_\Lambda : T^*M \to TM, \quad \sharp_\Lambda(\alpha_x) = i_{\alpha_x}\Lambda$$

$D = \mathrm{Im}\sharp_\Lambda$ integrable distribution whose leaves are symplectic manifolds

$(T^*M, [\![\cdot,\cdot]\!], \sharp_\Lambda)$ Lie algebroid

$$[\![\alpha,\beta]\!] = \mathcal{L}_{\sharp_\Lambda(\alpha)}\beta - \mathcal{L}_{\sharp_\Lambda(\beta)}\alpha - d(\beta(\sharp_\Lambda(\alpha)))$$

$$\Lambda \text{ Poisson structure} \Leftrightarrow \begin{cases} \sharp_\Lambda : T^* \to TM \text{ vector bundle homomorphism} \\ \sharp_\Lambda([\![\alpha,\beta]\!]) = [\sharp_\Lambda(\alpha), \sharp_\Lambda(\beta)] \quad \forall \alpha, \beta \in \Omega^1(M) \end{cases}$$

$\bar{\omega}$ k-poly-symplectic structure on M

$$\Updownarrow$$

$$\flat_{\overline{\omega}} : TM \to \flat_{\overline{\omega}}(TM) \subseteq (T^1_k)^* M := T^*M \oplus \stackrel{k}{\ldots} \oplus T^*M \quad \flat_{\overline{\omega}}(v_x) = (i_{v_x}(\omega^1_x), \ldots, i_{v_x}(\omega^k_x))$$

isomorphism vector bundle from TM to a sub-bundle of $T^*M \oplus \stackrel{k}{\ldots} \oplus T^*M$

- *Skew-symmetry*:

$$\flat_{\overline{\omega}}(v_x)(v_x) = (0, \ldots, 0) \qquad (v_x \in T_xM, \; x \in M).$$

- *Non degenerate condition*: $\flat_{\overline{\omega}}$ vector bundle monomorphism

$$\mathrm{Ker}\, (\flat_{\overline{\omega}}) = \{0\}$$

- *Integrability condition*:

$$\flat_{\overline{\omega}}([X, Y]) = \mathcal{L}_X \flat_{\overline{\omega}}(Y) - \mathcal{L}_Y \flat_{\overline{\omega}}(X) - \mathrm{d}(\flat_{\overline{\omega}}(X)(Y)),$$

$X, Y \in \mathfrak{X}(M)$

$(S, \sharp)$ poli-Poisson structure

$$\Updownarrow$$

S vector sub-bundle of $(T^1_k)^* M$

$\sharp : S \to TM$ vector bundle morphism

- *Skew-symmetry*: $\alpha_i(\sharp(\overline{\alpha})) = 0, \quad \overline{\alpha} = (\alpha_1, \ldots, \alpha_k),\ i \in \{1, \ldots, k\}$
- *Non degenerate condition*: If $\overline{\alpha}(\sharp(\overline{\beta})) = 0\ \forall \overline{\beta} \in S \Rightarrow \sharp(\overline{\alpha}) = 0$
- *Integrability condition*: $\overline{\alpha}, \overline{\beta} \in \Gamma(S)$

$$[\sharp(\overline{\alpha}), \sharp(\overline{\beta})] = \sharp\left(\mathcal{L}_{\sharp(\overline{\alpha})}\overline{\beta} - \mathcal{L}_{\sharp(\overline{\beta})}\overline{\alpha} - \mathrm{d}(\overline{\beta}(\sharp(\overline{\alpha})))\right)$$

- Polisymplectic manifold $\Rightarrow$ Poli-Poisson manifold

$$S = \flat_{\overline{\omega}}(TM) \text{ and } \sharp = \flat_{\overline{\omega}}^{-1}$$

- Poli-Poisson manifold of order $k = 1 \equiv$ Poisson manifold

PROPERTIES:

- $(M, S, \sharp)$ poli-Poisson manifold$\Rightarrow$ $D = \mathrm{Im}\sharp$ is integrable
- L a leaf of $\mathcal{F}_D \Rightarrow$ polisymplectic structure

INGREDIENTS:

- $(M, \bar{\omega} = (\omega_1, \ldots, \omega_k))$ a poly-simplectic manifold
- $\Phi : G \times M \to M$ a proper and free action such that $\Phi_g^* \omega_i = \omega_i, \ \ \forall g \in G$

$$\pi : M \to M/G$$

- $\mathrm{Im}\ (\flat_{\bar{\omega}}) \cap [(V\pi)^{\circ} \times \overset{(k}{\ldots} \times (V\pi)^{\circ}]$ is a vector sub-bundle
- $(\flat_{\bar{\omega}})^{-1} \left([((V\pi)^{\perp})^{\circ} \times \overset{(k}{\ldots} \times ((V\pi)^{\perp})^{\circ}] \cap [(V\pi)^{\circ} \times \overset{(k}{\ldots} \times \cap (V\pi)^{\circ}] \cap \mathrm{Im}\ (\flat_{\bar{\omega}}) \right) \subset V\pi$

$$(V\pi)^{\perp} = \bigcap_{i=1}^{k} (\omega_i^{\flat})^{-1}((V\pi)^{\circ})$$

$$\Downarrow$$

M/G is a poli-Poisson manifold

D. Iglesias, J.C. Marrero, M. Vaquero: On poly-Poisson structures, work in progress

G Lie group
G acts on the poly-symplectic manifold $(T^1_k)^*G$ and satisfies the above conditions

$$((T^1_k)^*G)/G \cong \mathfrak{g}^* \times \stackrel{k)}{\ldots} \times \mathfrak{g}^*$$

the leaves of the poly-symplectic foliation are the k-coadjoint orbits

J.C. Marrero, N. Román Roy, M. Salgado, S. Vilarino: Reduction of polisymplectic structures, work in progress

NEW OBJECTIVES

- To introduce the notion of multi-Poisson structure. Application to the reduction of first order classical field theories.
- To develop the multisymplectic reduction. Application to the reduction of first order classical field theories

Thanks!!!!!