An extension of the Marsden-Weinstein reduction process to the symplectic Lie algebroid setting

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Autonomous hamiltonian systems

- The configuration space: Q
- The phase space of momenta: T^*Q
- Hamiltonian function: $H : T^*Q \to \mathbb{R}$
- Structure: The canonical symplectic structure $\Omega_Q \in \Omega^2(T^*Q)$
- Hamiltonian vector field: $\mathcal{H}_H^{\Omega_Q} \in \mathfrak{X}(T^*Q)$, $\quad i_{\mathcal{H}_H^{\Omega_Q}} \Omega_Q = dH$
- Solutions of Hamilton equations: integral curves of $\mathcal{H}_H^{\Omega_Q}$

In the presence of a symmetry group

$\Downarrow$

The system can be reduced to a system with less freedom degrees

Reduction and reconstruction processes $\Rightarrow$ the solutions of Hamilton equations

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}, \qquad \frac{dq}{dt} = \frac{\partial H}{\partial p}$$

$$\{\cdot,\cdot\} : C^\infty(T^*Q) \times C^\infty(T^*Q) \to C^\infty(T^*Q)$$

The bracket of two linear functions with respect to $T^*Q \to Q$ is again linear

INGREDIENTS

- Poisson manifold $(M, \{\cdot,\cdot\})$
- $\phi : G \times M \to M$ a free and proper canonical Poisson action

$\Downarrow$ Reduction process

M/G a Poisson manifold

$$\{f_1, f_2\}_r \circ \pi = \{f_1 \circ \pi, f_2 \circ \pi\},$$

where $\pi : M \to M/G$ is the canonical projection and $f_1, f_2 \in C^\infty(M/G)$

$M = T^*Q \quad T^*\phi : G \times T^*Q \to T^*Q$ with $\phi : G \times Q \to Q$ free and proper action

$$T^*Q/G \to Q/G \text{ Poisson manifold}$$

$$\{\cdot,\cdot\} : C^\infty(T^*Q/G) \times C^\infty(T^*Q/G) \to C^\infty(T^*Q/G)$$

The bracket of two linear functions with respect to $T^*Q/G \to Q/G$ is again linear

$$Q = G \text{ and } \phi = l : G \times G \to G$$

$$T^*G \cong G \times \mathfrak{g}^* \Longrightarrow T^*G/G \cong (G \times \mathfrak{g}^*)/G \cong \mathfrak{g}^*$$

$$\Downarrow$$

$\mathfrak{g}^*$ with Lie-Poisson structure $\Lambda_{\mathfrak{g}^*}$

$$\Lambda_{\mathfrak{g}^*} = \sum c_{ij}^k \mu_k \frac{\partial}{\partial \mu_i} \wedge \frac{\partial}{\partial \mu_j}$$

$$T^*Q \to Q \qquad T^*Q/G \to Q/G$$

$\tau : A \to M$ vector bundle with linear Poisson structure on $A^* \to M$

$$\Downarrow$$

the bracket $\{\cdot,\cdot\}_{A^*}$ of two linear functions with respect to $A^* \to M$ is again linear

$$\Downarrow$$

- the bracket of a linear function and a basic function $f \circ \tau$ is a basic function
- the bracket of two basic function is zero

$$\{\hat{X} : A^* \to \mathbb{R}/\hat{X} \text{ is linear}\} \Longleftrightarrow \Gamma(A) = \{X : M \to A/X \text{ is a section of } A \to M\}$$

- $[\![\cdot,\cdot]\!] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ Lie bracket

$$\widehat{[\![X,Y]\!]} = -\{\hat{X},\hat{Y}\}_{A^*}$$

- $\rho : A \to TM$ vector bundle morphism (anchor map)

$$\rho(X)(f) \circ \pi = \{\hat{X}, f \circ \pi\}$$

$$[\![X, fY]\!] = f[\![X,Y]\!] + \rho(X)(f)Y, \quad \forall X,Y \in \Gamma(A), \quad \forall f \in C^\infty(M)$$

$$\{\Lambda_{A^*} \in \mathcal{V}^2(A^*) \text{ linear Poisson structure on } A^*\} \leftrightarrow \{([\![\cdot,\cdot]\!],\rho) \text{ Lie algebroid structure on } A\}$$

INGREDIENTS

- $(A \to M, [\![\cdot,\cdot]\!], \rho)$ Lie algebroid
- $\Phi : G \times A \to A$ action by complete lifts with base action $\phi : G \times M \to M$

$\psi : \mathfrak{g} \to \Gamma(A)$ Lie algebra anti-morphism such that $\Phi^* : G \times A^* \to A^*$ action by Poisson automorphisms

$$\xi_{A^*} = \mathcal{H}^{\Lambda_{A^*}}_{\widehat{\psi(\xi)}}$$

$$\Downarrow$$

$\Phi^T : TG \times A \to A$ an affine action of $TG \cong G \times \mathfrak{g}$ over A

$$\Phi^T((g,\xi), a_x) = \Phi_g(a_x + \psi(\xi)(x))$$

$\Downarrow$ Reduction process

$A/TG \to M/G$ is a Lie algebroid and $\widetilde{\pi} : A \to A/TG$ is a Lie algebroid epimorphism

The Lie algebroid

- *The vector bundle:* $\mathfrak{g} \times TM \to M \Longrightarrow \Gamma(\mathfrak{g} \times TM) \cong C^\infty(M, \mathfrak{g}) \times \mathfrak{X}(M)$
- *The Lie bracket:*

$$[(\xi_1, X_1), (\xi_2, X_2)] = ([\xi_1, \xi_2]_{\mathfrak{g}}, [X_1, X_2])$$

- *The anchor map:* $\rho(\xi, X) = X$

$\xi_1, \xi_2, \xi \in \mathfrak{g}$ and $X_1, X_2, X \in \mathfrak{X}(M)$.

The action by complete lifts

- *The action:* $\Phi : G \times (\mathfrak{g} \times TM) \to \mathfrak{g} \times TM$

$$\Phi_g(\xi, v_x) = (Ad^G_g \xi, T_x\phi_g(v_x))$$

  with $\phi : G \times M \to M$ free and proper action and $Ad^G : G \times \mathfrak{g} \to \mathfrak{g}$ is the adjoint action of G on $\mathfrak{g}$
- *The Lie algebra antimorphism:* $\psi : \mathfrak{g} \to \Gamma(\mathfrak{g} \times TM) \cong C^\infty(M, \mathfrak{g}) \times \mathfrak{X}(M)$

$$\psi(\bar{\xi}) = (-\bar{\xi}, \bar{\xi}_M), \quad \bar{\xi} \in \mathfrak{g},$$

- *The action* $\Phi^T : (G \times \mathfrak{g}) \times (\mathfrak{g} \times TM) \to \mathfrak{g} \times TM$

$$\Phi^T((g,\bar{\xi}),(\xi,v_x)) = (Ad_g^G(\xi - \bar{\xi}), T_x\phi_g(v_x + \bar{\xi}_M(x)))$$

- *The reduced Lie algebroid*

$$(\mathfrak{g} \times TM)/TG \cong (\mathfrak{g} \times TM)/(\mathfrak{g} \times G) \cong TM/G \to M/G$$

the Atiyah Lie algebroid induced by the principal bundle $\pi : M \to M/G$

$\tau : TM \to M$ is equivariant with respect to $T\phi : G \times TM \to TM$ and $\phi : G \times M \to M$

- *the vector bundle:* $\tau/G : TM/G \to M/G \Longrightarrow \Gamma(TM/G) \cong \mathfrak{X}(M)^G$
- *The bracket:* G-invariant vector fields are closed with respect to the Lie bracket of vector fields

$$[\![X, Y]\!]_{TM/G} = [X, Y]$$

- *The anchor:* $\rho_{TM/G} : TM/G \to T(M/G)$

$$\rho_{TM/G}(X(x)) = T_x\pi(X(x)),$$

X, Y G-invariant vector fields of M and $x \in M$

Marsden-Weinstein reduction Theorem

INGREDIENTS:

- (M, Ω) symplectic manifold
- $\phi : G \times M \to M$ a free and proper symplectic action of a Lie group G
- $J : M \to \mathfrak{g}^*$ an Ad^*-equivariant momentum map associated with ϕ

$$\xi_M = \mathcal{H}^{\Omega}_{J_\xi}, \quad J(\phi_g(x)) = \mathrm{Ad}^*_{g^{-1}}(J(x)), \qquad \text{for any } \xi \in \mathfrak{g}, x \in M$$

- $\nu \in \mathfrak{g}^*$

$$\phi_\nu : G_\nu \times J^{-1}(\nu) \to J^{-1}(\nu), \quad G_\nu = \{g \in G / \mathrm{Ad}^*_{g^{-1}} \nu = \nu\}$$

$$\Downarrow$$

$(J^{-1}(\nu)/G_\nu, \Omega_\nu)$ is a symplectic manifold

$$\pi^*_\nu \Omega_\nu = i^*_\nu \Omega$$

$$\pi_\nu : J^{-1}(\nu) \to J^{-1}(\nu)/G_\nu, \quad i_\nu : J^{-1}(\nu) \to M$$

- $(M, \Omega) = (T^*Q, \Omega_Q)$
- *The action:* $\phi : G \times Q \to Q$ a free and proper action

$$\Downarrow$$

$$T^*\phi : G \times T^*Q \to T^*Q \text{ free and proper symplectic action}$$

- *The momentum map:* $J : T^*Q \to \mathfrak{g}^*$ Ad^*-equivariant hamiltonian momentum map

$$J(\alpha_q)(\xi) = \alpha_q(\xi_Q(q))$$

- $\nu \in \mathfrak{g}^*$

$$\Downarrow \text{Reduction process}$$

$$(J^{-1}(\nu)/G_\nu, (\Omega_Q)_\nu) \text{ reduced symplectic manifold}$$

- If $\nu = 0$ then $(J^{-1}(\nu)/G_\nu, (\Omega_Q)_\nu) \cong (T^*(Q/G), \Omega_{Q/G})$
- If $G_\nu = G$ and

$$\exists \alpha_\nu \in \Omega^1(Q) \; G_\nu\text{-invariant 1-form with values in } J^{-1}(\nu)$$

$$\Downarrow$$

$$(J^{-1}(\nu)/G_\nu, (\Omega_Q)_\nu) \cong (T^*(Q/G), \Omega_{Q/G} - B_{\alpha_\nu})$$

- In other cases: symplectic embedding

$(A, [\![\cdot,\cdot]\!], \rho)$ Lie algebroid

- The differential of A $d^A : \Gamma(\wedge^k A^*) \to \Gamma(\wedge^k A^*), \quad (d^A)^2 = 0$

$$d^A f(X) = \rho(X)(f), \quad d^A(\alpha)(X,Y) = \rho(X)(\alpha(Y)) - \rho(Y)(\alpha(X)) - \alpha([\![X,Y]\!])$$

Symplectic-like Lie algebroid

$(A \to M, [\![\cdot,\cdot]\!], \rho)$ Lie algebroid of rank $2n + \Omega \in \Gamma(\wedge^2 A^*)$

$d^A\Omega = 0, \quad \flat_\Omega : A \to A^* \;\; \flat_\Omega(X) = i_X\Omega$ isomorphism of vector bundles

$$\Downarrow$$

M Poisson manifold

$$\{f_1, f_2\}_M = \Omega(\flat_\Omega^{-1}(d^A f_1), \flat_\Omega^{-1}(d^A f_2))$$

An example of a symplectic-like Lie algebroid: The cover of a fiberwise linear Poisson manifold

The Lie algebroid:

- *The vector bundle*

$$(\tau : A \to M, [\![\cdot,\cdot]\!], \rho) \text{ Lie algebroid of rank n}$$

$\mathcal{T}^A A^* \to A^*$ vector bundle

$$(\mathcal{T}^A A^*)_{a^*} = \{(a, v) \in A_{\tau^*(a^*)} \times T_{a^*} A^* / \rho(a) = T_{a^*}\tau^*(v)\}$$

- *The sections of* $\mathcal{T}^A A^* \to A^*$: generated by $(X, Y) \in \Gamma(A) \times \mathfrak{X}(A^*)$ such that $\rho(X) = T\tau^*(Y)$
- *The bracket*:

$$[\![(X_1, Y_1), (X_2, Y_2)]\!] = ([\![X_1, X_2]\!], [Y_1, Y_2])$$

- *The anchor map:*

$$\rho(X, Y) = Y$$

The symplectic-like structure:

$$\lambda_A \in \Gamma((\mathcal{T}^A A^*)^*), \quad \lambda_A(\alpha_x)(a_x, X_{\alpha_x}) = \alpha_x(a_x)$$

$$\Downarrow$$

$$\Omega_A = -d^{\mathcal{T}^A A^*} \lambda_A$$

INGREDIENTS

- $(A \to M, [\![\cdot,\cdot]\!], \rho, \Omega)$ a symplectic-like Lie algebroid
- $\Phi : G \times A \to A$ action by complete lifts with antimorphism $\psi : \mathfrak{g} \to \Gamma(A)$

$$\Phi_g^*(\Omega) = \Omega \text{ for all } g \in G$$

$$\Downarrow$$

$$\phi : G \times M \to M \text{ is a canonical Poisson action}$$

- $J : M \to \mathfrak{g}^*$ momentum map

$$i_{\psi(\xi)}\Omega = d^A J_\xi, \text{ and } Ad^*_{g-1}(J(x)) = J(\phi_g(x)), \quad \forall x \in M, \quad \forall g \in G \text{ and } \xi \in \mathfrak{g}$$

$$\Downarrow$$

$J : M \to \mathfrak{g}^*$ momentum map with respect to the Poisson manifold M

$$\Downarrow$$

$$J^T : A \to \mathfrak{g}^* \times \mathfrak{g}^*, \quad J^T(a) = ((TJ \circ \rho)(a), J(\tau(a)))$$

- It is equivariant with respect to $TG \times A \to A$
- If $\nu \in \mathfrak{g}^*$, $(J^T)^{-1}(0, \nu)$ is a subalgebroid of A
- $G_\nu \times (J^T)^{-1}(0, \nu) \to (J^T)^{-1}(0, \nu)$ action by complete lifts with respect

$$\psi : \mathfrak{g}_\nu \to \Gamma((J^T)^{-1}(0, \nu))$$

$$\Downarrow$$

$$TG_\mu \times (J^T)^{-1}(0, \nu) \to (J^T)^{-1}(0, \nu)$$

$$\Downarrow$$

$(A_\nu = (J^T)^{-1}(0, \nu)/TG_\nu \to J^{-1}(\nu)/G_\nu, \Omega_\nu)$ symplectic-like Lie algebroid

INGREDIENTS

- $(\mathcal{T}^A A^* \to A^*, \Omega_A)$ symplectic Lie algebroid
- $\Phi : G \times A \to A$ action by complete lifts with respect to $\psi : \mathfrak{g} \to \Gamma(A)$

$$\Downarrow$$

$$(\Phi, T\Phi^*) : G \times \mathcal{T}^A A^* \to \mathcal{T}^A A^*$$

$$\psi^T : \mathfrak{g} \to \Gamma(\mathcal{T}^A A^*)$$

- $J_{A^*} : A^* \to \mathfrak{g}^*$ hamiltonian momentum map

$$J_{A^*}(\alpha_x)(\xi) = \alpha_x(\psi(\xi)(x))$$

$$\Downarrow$$

$(\mathcal{T}^A A^*)_\nu = (J^T_{A^*})^{-1}(0,\nu)/TG_\nu \to J^{-1}_{A^*}(\nu)/G_\nu, \Omega_\nu)$ symplectic-like Lie algebroid

DESCRIPTION OF $((\mathcal{T}^A A^*)_\nu, \Omega_\nu)$

- If $\nu = 0$

$((\mathcal{T}^A A^*)_\nu, \Omega_\nu)$ is $\mathcal{T}^{A_0} A_0^*$ the canonical cover of the fiberwise linear Poisson

on the dual of $A_0 = A/TG \to M/G$ with the canonical symplectic section

- If $G_\nu = G$ and exists $\alpha_\nu \in \Gamma(A^*)$ G-invariant and with values in $J_{A^*}^{-1}(\nu)$

$((\mathcal{T}^A A^*)_\nu, \Omega_\nu)$ is $\mathcal{T}^{A_0} A_0^*$ the canonical cover of the fiberwise Poisson structure

on the dual of $A_0 = A/TG \to M/G$ with the symplectic section

deformed by a magnetic term $\Omega_{A_0} - (pr_1)^* B_\nu$

$$\beta_\nu = d^A \alpha_\nu \text{ is } TG_\nu\text{-invariant}$$

$$\Downarrow$$

$$\exists B_\nu \in \Gamma(\wedge^2 A_0^*) \text{ such that } \widetilde{\pi}^*(B_\nu) = \beta_\mu, \quad \widetilde{\pi} : A \to A_0 = A/TG$$

the magnetic term $(pr_1)^* B_\nu$, $\quad pr_1 : \mathcal{T}^{A_0} A_0^* \to A_0$

- Other cases: There is a canonical embedding

The example

$$A = \mathfrak{g} \times TM \to M$$

$\mathcal{T}^A A^* \to A^* \cong \mathfrak{g} \times T(\mathfrak{g}^* \times T^*M) \to \mathfrak{g}^* \times T^*M$

$(\mathfrak{g} \times TM)/TG \to M/G$ is isomorphic to the Atiyah algebroid associated with the principal bundle $\pi_M : M \to M/G$.

1. $\nu = 0$ The reduced Lie algebroid is isomorphic to the Atiyah algebroid associated with the principal bundle $\pi_{T^*M} : T^*M \to (T^*M)/G$ and its symplectic-like structure $\Omega_{(T^*M)/G} \in \Gamma(\wedge^2(T^*(T^*M)/G))$ is the one induced by the G-invariant symplectic structure on T^*M

2. If $G_\nu = G$ then the reduced Lie algebroid is simplectically isomorphic to the Atiyah algebroid associated with the principal bundle $\pi_{T^*M} : T^*M \to (T^*M)/G$ endowed with the symplectic-like structure

$$\Omega_{(T^*M)/G} - \gamma_\mu$$

where $\gamma_\mu \in \Gamma(\wedge^2(T^*M/G))$ is the 2-section obtained from a magnetic term

Conclusion

The problem of developing an extension of reduction process for (symplectic-like) Lie algebroids has been studied, and in particular to the case of the canonical cover of a fiberwise linear Poisson structure, whose reduction process is the analogue to cotangent bundle reduction in the context of Lie algebroids. In this case, we have described the reduced space like a canonical cover of a fiberwise linear Poisson structure whose symplectic section is deformed by a magnetic term

NEW OBJECTIVES:

- To apply this reduction process to new examples
- Bundle version reduction of the symplectic cover of a fiberwise linear Poisson manifold
- Reduction by stages of symplectic-like Lie algebroids

Thanks for your attention!!!

We are the best (for the football!!!)