UNIVERSITÀ DEGLI STUDI DI BARI

Dottorato di Ricerca in Matematica

XXV Ciclo – A.A. 2012/2013

Settore Scientifico-Disciplinare:

MAT/03 – Geometria

Tesi di Dottorato

Symplectic principal $\mathbb{R}$-bundle reduction and applications to non-autonomous Hamiltonian systems

Candidato:

Ignazio Lacirasella

Supervisori della tesi:

Prof. J.C. Marrero González

Prof.ssa A.M. Pastore

Coordinatore del Dottorato di Ricerca:

Prof.ssa A. SALVATORE

Tesi di Dottorato

Symplectic principal $\mathbb{R}$-bundle reduction and applications to non-autonomous Hamiltonian systems

Ignazio Lacirasella

Tesi di Dottorato realizzata in cotutela tra l'Università degli Studi di Bari Aldo Moro e l'Universidad de La Laguna, sotto la supervisione della Prof.ssa Anna Maria Pastore e del Prof. Juan Carlos Marrero

Tesis de Doctorado realizada en cotutela entre la Università degli Studi di Bari Aldo Moro y la Universidad de La Laguna, bajo la dirección de la Prof.ra Anna Maria Pastore y del Prof. Juan Carlos Marrero

Bari, febbraio 2013

Acknowledgments

My first acknowledgment is for Prof. Anna Maria Pastore, my supervisor at Bari. I began with her this PhD program, in which I had a lot of awesome experiences.

My deepest gratitude is for Prof. Juan Carlos Marrero. He has been a great supervisor and a great teacher; without him nothing of this thesis could have been written. A special mention for Prof. Edith Padrón. Thanks to her for the constant support and the great hospitality during my period of research at the Department of Mathematics of University of La Laguna.

A particular thank to my best mathematical friends, Lucio, Marco and Fabio. Thanks for your precious friendship, for our funny conversations, for sharing your life with me, the work was not so heavy with you.

My time in Spain was made enjoyable in large part due to the many friends that became a part of my life. Thanks to Mònica, Miguel, Elisa, and David.

I would also like to thank all the friends and collegues at the University of Bari and all the members of the Geometry, Mechanics and Control Network. A list of you would be too long. Thanks to everyone.

At last, but not at least, my greatest thanks to my family which always sustains and encourages me and, of course, to Angela. Everyone who knows me also knows how important her support and infinite patience is for me.

For this dissertation, I would like to thank the referees of the present work, Prof. David Martín de Diego of ICMAT (Madrid) and Prof. Frans Cantrijn of Ghent University.

The PhD program has been supported by University of Bari, *Dottorato di Ricerca in Matematica, XXV Ciclo*. Moreover, it has been partially supported by MEC (Spain) grants *Agencia Canaria de Investigación, Innovación y Sociedad de la Informacción, Gobierno de Canarias, ProlD20100210, Ministerio de Ciencia e Innovación MTM2009-13383, MTM2010-12116-E* and *Secretaría de Estado de Investigación, Desarrollo e Innovación MTM2012-34478, MTM2011-15725-E.*

Contenuto della Tesi

La teoria della *riduzione* nasce agli albori della Meccanica con gli studi di Eulero, Lagrange, Routh, Riemann e Poincaré. Lo scopo dei loro lavori era "eliminare le variabili" associate alle simmetrie al fine di semplificare i calcoli in esempi concreti. La formulazione moderna della riduzione simplettica fu sviluppata da Marsden e Weinstein in un lavoro sulla riduzione di varietà simplettiche in presenza di un'applicazione momento [49]. L'idea principale di questo tipo di riduzione è la seguente: supponiamo che un gruppo di Lie agisca simpletticamente su una varietà simplettica in presenza di un'applicazione momento. L'insieme di livello dell'applicazione momento è munito canonicamente di una 2-forma chiusa che, in generale, è degenere. Per eliminare questa degenerazione, si considera lo spazio delle orbite dell'azione di un certo gruppo di isotropia sull'insieme di livello. In questo modo si ottiene una struttura simplettica sullo spazio ridotto.

In Geometria Differenziale sono note numerose applicazioni della teoria della riduzione. Il primo esempio è una costruzione alternativa della classica struttura simplettica di Kirillov-Kostant-Souriau sull'orbita dell'azione coaggiunta di gruppo di Lie (cfr. [1]). Inoltre, idee simili sulla riduzione di strutture geometriche con applicazioni momento sono state usate per ridurre strutture di Poisson, cosimplettiche, Kähler, hyperkähler, contatto, f-structure, etc. ([2, 17, 18, 25, 26]).

In Meccanica Geometrica, le varietà simplettiche sono di particolare interesse, in quanto descrivono la dinamica di un sistema meccanico governato da una funzione Hamiltoniana. Infatti, lo spazio delle fasi è rappresentato dal fibrato cotangente T^*Q di uno spazio delle configurazioni Q. T^*Q è munito canonicamente di una struttura simplettica che ci permette di descrivere la equazioni del moto in forma intrinseca. Se un gruppo di Lie agisce su Q, si può ridurre la varietà simplettica T^*Q rispetto al sollevamento cotangente dell'azione e a un'applicazione momento canonica. In questa situazione, ci si potrebbe chiedere se la varietà simplettica ridotta è nuovamente un fibrato cotangente con la sua struttura simplettica canonica. La risposta a tale domanda si può trovare nella cosiddetta *cotangent bundle reduction theory* (cfr. ad esempio [34, 41, 42, 53]).

Come abbiamo poc'anzi detto, le varietà simplettiche (e, in particolare, il fibrato cotangente T^*Q di una varietà Q) sono gli spazi naturali sui quali si sviluppa la formulazione della Meccanica Classica nel caso autonomo (cfr. ad esempio, [1, 16, 37]). Nel contesto non-autonomo, la situazione è differente. Lo spazio delle configurazioni in questo caso è dato da un varietà che fibra sulla retta reale, cioè è dato da una fibrazione $\pi : M \to \mathbb{R}$, rispetto alla quale lo spazio delle fasi ristretto è rappresentato dal duale $V^*\pi$ del fibrato verticale $V\pi$ di π, mentre lo spazio delle fasi esteso è dato dal fibrato cotangente T^*M di M (cfr. ad esempio [14] e riferimenti bibliografici in esso contenuti; cfr. anche [15, 20, 27, 51]).

In questo contesto, una funzione Hamiltoniana su un opportuno spazio delle fasi non è sufficiente a descrivere la dinamica del sistema. Infatti, nonostante lo spazio ristretto delle fasi $V^*\pi$ è munito canonicamente di una struttura di Poisson, i campi Hamiltoniani per tale struttura sono verticali rispetto alla proiezione sul tempo. Pertanto, non possono descrivere la dinamica. Nel caso in cui la fibrazione $\pi : M \to \mathbb{R}$ sia triviale, cioè, se $M = Q \times \mathbb{R}$, si aggiunge un campo vettoriale ∂_t su $V^*\pi$ al fine di ottenere la dinamica. Se, invece, la fibrazione π non è triviale, è necessario fissare un campo vettoriale che si proietta su ∂_t e che rappresenta la scelta di un sistema di riferimento. Cambiando il sistema di riferimento, cambia anche la funzione Hamiltoniana. Per avere una descrizione della dinamica che sia indipendente dal sistema di riferimento, è necessario sostituire la nozione di funzione Hamiltoniana con quella di *sezione* Hamiltoniana $h : V^*\pi \to T^*M$ della proiezione canonica $\mu_\pi : T^*M \to V^*\pi$. In questo modo, usando la struttura simplettica canonica su T^*M (rispettivamente, un'opportuna struttura cosimplettica su $V^*\pi$), si può sviluppare il formalismo Hamiltoniano esteso (rispettivamente, ristretto). Per ulteriori dettagli, cfr. il paragrafo 1.3.3 e [14, 20, 27].

Osserviamo che $\mu_\pi : T^*M \to V^*\pi$ è un $\mathbb{R}$-fibrato principale (un *AV-fibrato* nella terminologia di [20], dove AV sta per *Affine Values*), la cui $\mathbb{R}$-azione principale (affine) $\psi_\pi : \mathbb{R} \times T^*M \to T^*M$ è data da

$$\psi_\pi(s, \alpha_x) = \alpha_x + s(\pi^*(dt))(x), \qquad \text{per ogni } s \in \mathbb{R},\ \alpha_x \in T_x^*M,$$

dove $\pi^*(dt)$ è il pull-back della 1-forma globale dt su $\mathbb{R}$ e t è la coordinata globale standard su $\mathbb{R}$. Inoltre, l'azione principale è simplettica. Questo ci suggerisce di introdurre la nozione di *symplectic principal* $\mathbb{R}$*-bundle* ($\mathbb{R}$-fibrato principale simplettico) come un $\mathbb{R}$-fibrato principale con spazio totale simplettico e con azione principale simplettica.

Inoltre, in maniera naturale, introduciamo la nozione di *sistema Hamiltoniano non-autonomo* come un $\mathbb{R}$-fibrato principale simplettico $\mu : A \to V$ e una sezione Hamiltoniana $h : V \to A$, cioè una sezione della proiezione μ. La sezione Hamiltoniana h induce un campo vettoriale su V le cui curve integrali sono le soluzioni delle equazioni della dinamica per il sistema Hamiltoniano (cfr. paragrafo 4.1). Nel caso particolare in cui $\mu : A \to V$ sia l'$\mathbb{R}$-fibrato principale simplettico *standard* (cioè $\mu = \mu_\pi$ con $\pi : M \to \mathbb{R}$ fibrazione su $\mathbb{R}$), otteniamo le classiche equazioni di Hamilton. Tutti gli strumenti usati nella teoria classica della Meccanica Geometrica non-autonoma, come il formalismo Lagrangiano e Hamiltoniano o la formulazione variazionale, sono presenti nel contesto degli AV-fibrati (cfr. [20, 21, 22, 23, 60]).

Precisiamo anche che la teoria degli AV-fibrati è relazionata con la teoria degli affgebroidi di Lie che rappresentano l'analogo affine degli algebroidi di Lie (per ulteriori dettagli su questo tema, cfr. [20, 24, 27, 51, 57]).

Lo scopo del presente lavoro di Tesi è sviluppare un processo di riduzione nel contesto degli $\mathbb{R}$-fibrati principali simplettici. Introduciamo la nozione di simmetria di $\mathbb{R}$-fibrato principale simplettico e mostriamo che, sotto ipotesi di regolarità e in presenza di un'applicazione momento, si può ottenere un $\mathbb{R}$-fibrato principale simplettico ridotto. Applichiamo tale processo di riduzione al $\mathbb{R}$-fibrato principale simplettico standard associato ad una fibrazione. Infine, proviamo che un sistema Hamiltoniano non autonomo con sezione Hamiltoniana equivariante induce una sezione Hamiltoniana sull'$\mathbb{R}$-fibrato principale simplettico ridotto.

Alcuni dei risultati del presente lavoro di Tesi sono contenuti in [35].

Struttura della tesi: Nel seguito, saranno elencati brevemente i contributi in Geometria Differenziale e in Meccanica Geometrica ottenuti in questa dissertazione.

Nel Capitolo 1, sono introdotti alcuni concetti fondamentali relativi alla formulazione geometrica della Meccanica. In particolare, le strutture di Poisson, simplettiche e cosimplettiche sono utilizzate per la descrizione dei sistemi meccanici autonomi e non autonomi. Si richiamano i processi di riduzione con applicazione momento per tali tipi di strutture nelle due versioni, *point version* e *orbit version*, descritte rispettivamente nei paragrafi 1.5.1 e 1.5.2. Nel paragrafo 1.5.2 diamo una versione per orbite della riduzione cosimplettica, che, in base alle nostre conoscenze, non era stata sviluppata in letteratura.

D'altra parte, se applichiamo il teorema di riduzione simplettica al fibrato cotangente di una varietà Q rispetto al sollevamento cotangente di un'azione su Q, sotto opportune ipotesi, si può ottenere nuovamente un fibrato cotangente, dove la struttura simplettica canonica è deformata da un termine magnetico. Questo risultato è descritto nella *versione embedding* della *cotangent bundle reduction* (cfr. pararafo 1.6.2). Inoltre, fissata una connessione principale, si può descrivere la varietà simplettica ridotta in termini della struttura simplettica canonica, della struttura simplettica di Kirillov-Kostant-Souriau su un'orbita coaggiunta e della curvatura della connessione fissata (cfr. paragrafo 1.6.3).

Nel Capitolo 2, introduciamo il concetto di $\mathbb{R}$-fibrato principale simplettico, motivato dallo studio dei sistemi meccanici Hamiltoniani non autonomi. Un $\mathbb{R}$-fibrato principale simplettico è una terna (A, μ, Ω), dove (A, Ω) è una varietà simplettica e $\mu\colon A \to V$ è un $\mathbb{R}$-fibrato principale la cui azione principale è simplettica. Innanzitutto, proviamo che lo spazio base V è canonicamente munito di una struttura di Poisson. Nel paragrafo 2.2, mostriamo l'esistenza di coordinate di Darboux adattate alla fibrazione principale e studiamo la categoria degli $\mathbb{R}$-fibrati principali simplettici. In particolare, descriviamo la relazione fra le strutture di Poisson indotte sui corrispondenti spazi base di un embedding di $\mathbb{R}$-fibrati principali simplettici.

Un'azione di un gruppo di Lie G sullo spazio totale A di un $\mathbb{R}$-fibrato principale simplettico è detta *canonica* se l'azione di ogni elemento $g \in G$ è un morfismo della struttura. Inoltre, imponiamo un'opportuna condizione di compatibilità con il generatore infinitesimo dell'azione principale (cfr. paragrafo 2.3). Quindi, nel paragrafo 2.4, descriviamo la nostra procedura di riduzione: in presenza di un'azione canonica e di un'applicazione momento e sotto opportune ipotesi di regolarità, si può ottenere un $\mathbb{R}$-fibrato principale simplettico ridotto. Lo spazio totale e lo spazio base del $\mathbb{R}$-fibrato ridotto sono spazi delle orbite dell'azione di un'opportuno gruppo di isotropia che agisce sugli insiemi di livello dell'applicazione momento (*point version*). Il paragrafo 2.5, invece, è dedicato alla *orbit version* della riduzione di $\mathbb{R}$-fibrati principali simplettici. In tale versione, lo spazio totale e lo spazio base del $\mathbb{R}$-fibrato ridotto sono spazi delle orbite dell'azione del gruppo di Lie che agisce sull'immagine reciproca di un'orbita coaggiunta, calcolata usando l'applicazione momento. Proviamo, inoltre, che le due versioni sono equivalenti, in quanto gli $\mathbb{R}$-fibrati principali simplettici ridotti risultanti sono isomorfi.

Un $\mathbb{R}$-fibrato principale simplettico è detto *standard* se esiste una fibrazione $\pi\colon M \to \mathbb{R}$ tale che la proiezione principale è l'applicazione canonica $\mu_\pi\colon T^*M \to V^*\pi$. Nel Capitolo 3, studiamo il processo di riduzione per questi fibrati: se π è invariante rispetto ad un'azione libera e propria di un gruppo di Lie G su M, allora il sollevamento cotangente di tale azione è un'azione canonica e possiamo applicare il processo di riduzione. Nel caso particolare in cui $\pi\colon M \to \mathbb{R}$ è un G-fibrato principale, otteniamo che lo spazio base del $\mathbb{R}$-fibrato principale simplettico ridotto è una varietà di Poisson di corango 1 le cui foglie simplettiche sono isomorfe ad un'orbita coaggiunta.

D'altra parte, nel caso generale e in analogia alla *cotangent bundle reduc-*

tion, ci poniamo la seguente questione: applicando il processo di riduzione ad un $\mathbb{R}$-fibrato principale simplettico standard si ottiene nuovamente una struttura standard? Nei paragrafi 3.3, 3.4 e 3.5 analizziamo questo problema. Otteniamo che, sotto opportune ipotesi, si può ottenere nuovamente un $\mathbb{R}$-fibrato principale simplettico standard, dove la struttura simplettica è deformata da un termine magnetico. In maniera analoga, otteniamo una "deformazione magnetica contravariante" del bivettore di Poisson sullo spazio base ridotto. Nel paragrafo 3.5 descriviamo tale deformazione come sollevamento verticale di un'opportuna 2-forma ottenuta dalla curvatura di una connessione principale.

Nel Capitolo 4, consideriamo una dinamica nell'$\mathbb{R}$-fibrato principale simplettico (A, μ, Ω), ovvero una sezione $h\colon V \to A$ di $\mu\colon A \to V$. Nel paragrafo 4.1, proviamo che una sezione Hamiltoniana può essere equivalentemente definita da una funzione $F_h : A \to \mathbb{R}$ omogenea di grado 1 rispetto al generatore infinitesimale dell'azione principale di $\mathbb{R}$ su A. Proviamo, inoltre, che una sezione Hamiltoniana induce una struttura cosimplettica sullo spazio base V. Il primo risultato di questo capitolo, che motiva lo studio di questo tipo di strutture, è che le curve integrali del corrispondente campo di Reeb sono le soluzioni delle equazioni di Hamilton definite da h. Pertanto, definiamo un *sistema Hamiltoniano non autonomo* (A, μ, Ω, h) come un $\mathbb{R}$-fibrato principale simplettico (A, μ, Ω) e una sezione Hamiltoniana $h\colon V \to A$. In seguito, nel paragrafo 4.2, studiamo i morfismi di queste strutture e proviamo che tali morfismi preservano la struttura cosimplettica sullo spazio base e, pertanto, anche la dinamica del sistema. Quindi, descriviamo un processo di riduzione per questo tipo di strutture. Se la sezione Hamiltoniana è invariante rispetto all'azione canonica, allora si può ottenere una sezione Hamiltoniana ridotta. Infine, nel paragrafo 4.4, studiamo la cosiddetta procedura di *ricostruzione*. Supponiamo che una soluzione delle equazioni di Hamilton ridotte siano note e descriviamo come si può ottenere il moto del sistema Hamiltoniano iniziale.

Il Capitolo 5 è dedicato allo studio di alcuni esempi che possono essere discussi nel contesto degli $\mathbb{R}$-fibrati principali simplettici. Innanzitutto, supponiamo che lo spazio delle configurazioni sia il prodotto di un gruppo di Lie G con la retta reale, cioè $M = G \times \mathbb{R}$. Nel paragrafo 5.1, studiamo un processo di riduzione per il corrispondente $\mathbb{R}$-fibrato principale simplettico standard μ_π. Fissata una metrica Riemanniana invariante a sinistra e una funzione potenziale su G (entrambe dipendenti dal tempo), si ottiene un particolare tipo di sezioni Hamiltoniane, denominate *di tipo meccanico*. Il gruppo delle simmetrie è un sottogruppo chiuso K di G tale che G ammetta una decomposizione riduttiva rispetto a K. Quindi, otteniamo una condizione sufficiente affinché il sistema Hamiltoniano non-autonomo ridotto sia nuovamente di tipo meccanico (cfr. Proposizione 5.3). Nei paragrafi 5.2 e 5.3 discutiamo due esempi espliciti: l'*heavy top* dipendente dal tempo ($G = SO(3)$) e l'oscillatore armonico smorzato dipendente dal tempo ($G = S^1 \times \mathbb{R}$). Infine, nel paragrafo 5.4, applichiamo i nostri processi di riduzione alla formulazione (indipendente dal sistema di riferimento) della Meccanica analitica nello spazio-tempo Newtoniano.

La Tesi di Dottorato si conclude con le nostre conclusioni e una descrizione delle direzioni di ricerca future.

Contenido de la Tesis

La teoría de *reducción* tiene sus antecedentes en los orígenes de la Mecánica con los estudios de Euler, Lagrange, Routh, Riemann y Poincaré. El objetivo de sus trabajos fue "eliminar variables" asociadas con simetrías con el fin de simplificar los cálculos en ejemplos concretos. La formulación moderna de la teoría de la reducción fue desarrollada por Marsden y Weinstein en un trabajo sobre reducción de variedades simplécticas en presencia de una aplicación momento ([49]). La principal idea de este tipo de reducción es la siguiente: supongamos que tenemos un grupo de Lie que actúa simplécticamente sobre una variedad simpléctica y una aplicación momento asociada a esta acción. El conjunto de nivel de la aplicación momento admite una 2-forma cerrada que, en general, es degenerada. Para eliminar esta degeneración, se considera el espacio de órbitas de la acción de un cierto grupo de isotropía sobre el conjunto de nivel. De esta manera, se obtiene finalmente una estructura simpléctica sobre el espacio reducido.

En Geometría Diferencial se conocen muchas aplicaciones de la teoría de reducción. El primer ejemplo es una construcción alternativa de la estructura simpléctica clásica de Kirillov-Kostant-Souriau sobre la órbita de la acción coadjunta de un grupo de Lie (veáse [1]). Además, ideas similares sobre la reducción de estructuras simplécticas con aplicación momento han sido utilizadas para reducir estructuras de Poisson, cosimplécticas, Kähler, hyperkähler, contacto, f-estructuras, etc. (veáse [2, 17, 18, 25, 26]).

En Mecánica Geométrica, las variedades simplécticas son de especial interés, porque describen la dinámica de un sistema mecánico gobernado por una función Hamiltoniana. De hecho, el espacio de fases del sistema está representado por el fibrado cotangente T^*Q del espacio de configuración Q. T^*Q está equipado canónicamente con una estructura simpléctica que nos permite describir las ecuaciones del movimiento de forma intrínseca. Si un grupo de Lie actúa sobre Q, uno podría reducir la variedad simpléctica T^*Q con respecto al levantamiento cotangente de la acción y a una aplicación momento canónica. En esta situación, uno podría preguntarse si la variedad simpléctica reducida es de nuevo un fibrado cotangente con su estructura simpléctica canónica. La respuesta a esta pregunta nos la proporciona lo que en la literatura actual se conoce con el nombre de *cotangent bundle reduction theory* (veáse [34, 41, 42, 53]).

Como mencionamos anteriormente, las variedades simplécticas (y, en particular, el fibrado cotangente T^*Q de una variedad Q) son los espacios naturales donde se desarrolla la formulación de la Mecánica Clásica en el caso autónomo (veáse, por ejemplo, [1, 16, 37]). En el contexto no-autónomo, la situación es diferente. En este caso, el espacio de configuración es una variedad que fibra sobre la recta real, es decir tenemos una fibración $\pi : M \to \mathbb{R}$, respecto de la cual el espacio de fases restringido está representado por el dual $V^*\pi$ del fibrado vertical $V\pi$ de π. El espacio de fases extendido es el fibrado cotangente T^*M de M (veáse, por ejemplo, [14] y referencias contenidas en este trabajo; veáse también [15, 20, 27, 51]).

En este contexto, una función Hamiltoniana sobre algún espacio de fases no es suficiente para describir la dinámica del sistema. A pesar del hecho que el espacio restringido de fases $V^*\pi$ está canónicamente equipado con una estructura de Poisson, los campos Hamiltonianos de esta estructura son verticales con respecto a la proyección sobre el tiempo. Por lo tanto, no pueden describir la dinámica. En el caso que la fibración sea trivial, es decir, si $M = Q \times \mathbb{R}$, se añade el campo de vectores ∂_t sobre $V^*\pi$ para obtener la dinámica. Si la fibración no es trivial, se necesita fijar un campo de vectores que se proyecta sobre ∂_t y que representa la elección de un sistema de referencia. Cambiando el sistema de referencia, cambia también la función Hamiltoniana. Para obtener una descripción de la dinámica que sea independiente del sistema de referencia, se necesita reemplazar la noción de función Hamiltoniana por la de *sección Hamiltoniana* $h : V^*\pi \to T^*M$ de la proyección canónica $\mu_\pi : T^*M \to V^*\pi$. De esta manera, utilizando la estructura simpléctica canónica sobre T^*M (respectivamente, una adecuada estructura cosimpléctica sobre $V^*\pi$), se puede desarrollar el formalismo Hamiltoniano extendido (respectivamente, restringido). Para más detalles, veáse la Sección 1.3.3 y [14, 20, 27].

Observamos que $\mu_\pi : T^*M \to V^*\pi$ es un $\mathbb{R}$-fibrado principal (un *AV-fibrado* en la terminología de [20], donde AV abrevia *Affine Values*) cuya $\mathbb{R}$-acción principal (afín) $\psi_\pi : \mathbb{R} \times T^*M \to T^*M$ está dada por

$$\psi_\pi(s, \alpha_x) = \alpha_x + s(\pi^*(dt))(x), \qquad \text{por cada } s \in \mathbb{R},\ \alpha_x \in T^*_xM,$$

donde $\pi^*(dt)$ es el pull-back de la 1-forma global dt sobre $\mathbb{R}$ y t es la coordenada global canónica sobre $\mathbb{R}$. Además, la acción principal es simpléctica. Esto nos sugiere introducir la noción de *symplectic principal* $\mathbb{R}$*-bundle* ($\mathbb{R}$-fibrado principal simpléctico) como un $\mathbb{R}$-fibrado principal con espacio total simpléctico y con acción principal simpléctica.

De manera natural introducimos también la noción de *sistema Hamiltoniano no-autónomo* como un $\mathbb{R}$-fibrado principal simpléctico $\mu : A \to V$ y una sección Hamiltoniana $h : V \to A$, es decir una sección de la proyección μ. La sección Hamiltoniana h induce un campo de vectores sobre V cuyas curvas integrales son las soluciones de las ecuaciones de la dinámica para el sistema Hamiltoniano (veáse Sección 4.1). En el caso particular en que μ sea el $\mathbb{R}$-fibrado principal simpléctico *estándar* (es decir $\mu = \mu_\pi$ con $\pi : M \to \mathbb{R}$ una fibración sobre $\mathbb{R}$), obtenemos las ecuaciones clásicas de Hamilton. Todas las herramientas utilizadas en la teoría clásica de la Mecánica Geométrica no-autónoma, como el formalismo Lagrangiano y Hamiltoniano o la formulación variacional, están presentes en el contexto de los AV-fibrados (veáse [20, 21, 22, 23, 60]).

La teoría de los AV-fibrados está relacionada con la teoría de los afgebroides de Lie que representan el análogo afín de los algebroides de Lie (para más detalles sobre este tema, veáse [20, 24, 27, 51, 57]).

El objetivo de esta Memoria es desarrollar el proceso de reducción en el contexto de los $\mathbb{R}$-fibrados principales simplécticos. Introducimos la noción de simetría de $\mathbb{R}$-fibrado principal simpléctico y mostramos que, bajo hipótesis de regularidad y en presencia de una aplicación momento, se puede obtener un $\mathbb{R}$-fibrado principal simpléctico reducido. Aplicamos este proceso de reducción al $\mathbb{R}$-fibrado principal simpléctico estándar asociado a una fibración. Finalmente, probamos que un sistema Hamiltoniano no-autónomo con una sección Hamiltoniana equivariante induce una sección Hamiltoniana sobre el $\mathbb{R}$-fibrado principal simpléctico reducido.

Algunos de los resultados de esta Memoria están contenidos en [35].

Estructura de la tesis: Precisaremos a continuación brevemente las contribuciones en Geometría Diferencial y en Mecánica Geométrica obtenidas en esta Memoria.

En el Capítulo 1, se introducen algunos conceptos fundamentales relacionados con la formulación geométrica de la Mecánica. En particular, las estructuras de Poisson, simplécticas y cosimplécticas se usan en la descripción de los sistemas mecánicos autónomos y no-autónomos. Recordamos los procesos de reducción con aplicación momento para estas estructuras en dos versiones, *point version* y *orbit version*, que se describen respectivamente en las Secciones 1.5.1 y 1.5.2. En la Sección 1.5.2 damos una versión por órbitas de la reducción cosimpléctica, que, hasta nuestro conocimiento, no había sido desarrollado en la literatura.

Por otro lado, si aplicamos el teorema de reducción simpléctica al fibrado cotangente de una variedad Q con respecto al levantamiento cotangente de una acción sobre Q, bajo adecuadas hipótesis, se puede obtener nuevamente un fibrado cotangente, donde la estructura simpléctica canónica está deformada por un término magnético. Éste resultado está descrito en la *versión embedding* de la *cotangent bundle reduction* (veáse Sección 1.6.2). Además, fijada una conexión principal, se puede describir la variedad simpléctica reducida en términos de la estructura simpléctica canónica, de la estructura simpléctica de Kirillov-Kostant-Souriau sobre una órbita coadjunta y de la curvatura de la conexión fijada (veáse Sección 1.6.3).

En el Capítulo 2, introducimos la noción de $\mathbb{R}$-fibrado principal simpléctico, motivado por el estudio de los sistemas mecánicos Hamiltonianos no-autónomos. Un $\mathbb{R}$-fibrado principal simpléctico es un triple (A, μ, Ω), donde (A, Ω) es una variedad simpléctica y $\mu\colon A \to V$ es un $\mathbb{R}$-fibrado principal cuya acción principal es simpléctica. En primer lugar, probamos que el espacio base V está canónicamente equipado con una estructura de Poisson. En la Sección 2.2, mostramos la existencia de coordenadas de Darboux adaptadas a la fibración principal y estudiamos la categoría de los $\mathbb{R}$-fibrados principales simplécticos. En particular, describimos la relación entre las estructuras de Poisson inducidas sobre los correspondientes espacios base de un embebimiento de $\mathbb{R}$-fibrados principales simplécticos.

Una acción de un grupo de Lie G sobre el espacio total de un $\mathbb{R}$-fibrado principal simpléctico A se dice *canónica* si la acción de cada elemento $g \in G$ es un morfismo de la estructura. Además, imponemos una adecuada condición de compatibilidad con el generador infinitesimal de la acción principal (veáse Sección 2.3). A continuación, en la Sección 2.4, describimos nuestro proceso de reducción: en presencia de una acción canónica y de una aplicación momento y bajo adecuadas hipótesis de regularidad, se puede obtener un $\mathbb{R}$-fibrado principal simpléctico reducido. El espacio total y el espacio base del $\mathbb{R}$-fibrado reducido son espacios de órbitas de la acción de un adecuado grupo de isotropía que actúa sobre los conjuntos de nivel de la aplicación momento (*point version*). Por otra parte, la Sección 2.5 está dedicada a la *orbit version* de la reducción de $\mathbb{R}$-fibrados principales simplécticos. En esta versión, el espacio total y el espacio base del $\mathbb{R}$-fibrado reducido son espacios de órbitas de la acción del grupo de Lie que actúa sobre la preimagen de una órbita coadjunta, calculada usando la aplicación momento. Además, probamos que las dos versiones son equivalentes, porque los $\mathbb{R}$-fibrados principales simplécticos reducidos resultantes son isomorfos.

Un $\mathbb{R}$-fibrado principal simpléctico se dice *estándar* si existe una fibración $\pi\colon M \to \mathbb{R}$ tal que la proyección principal está dada por la aplicación canónica $\mu_\pi\colon T^*M \to V^*\pi$. En el Capítulo 3, estudiamos el proceso de reducción para estos fibrados: si π es invariante respecto de una acción libre y propia de un grupo de Lie G sobre M, entonces el levantamiento cotangente de esta acción es una acción canónica y podemos aplicar el proceso de reducción. En el caso particular donde $\pi\colon M \to \mathbb{R}$ es un G-fibrado principal, obtenemos que el espacio base del $\mathbb{R}$-fibrado principal simpléctico reducido es una variedad de Poisson de corango 1 cuyas hojas simplécticas son isomorfas a una órbita coadjunta.

Por otra parte, en el caso general y de forma similar que en la *cotangent bundle reduction*, nos planteamos la siguiente cuestión: ¿aplicando el proceso de reducción a un $\mathbb{R}$-fibrado principal simpléctico estándar se obtiene nuevamente una estructura estándar? En las Secciones 3.3, 3.4 y 3.5 analizamos este problema. Obtenemos que, bajo hipótesis adecuadas, se puede obtener nuevamente un $\mathbb{R}$-fibrado principal simpléctico estándar, donde la estructura simpléctica está deformada por un término magnético. De manera similar, obtenemos una "deformación magnética contravariante" del bivector de Poisson sobre el espacio base reducido. En la Sección 3.5 describimos esta deformación como el levantamiento vertical de una adecuada 2-forma obtenida a partir de la curvatura de una conexión principal.

En el Capítulo 4, consideramos una dinámica en el $\mathbb{R}$-fibrado principal simpléctico (A, μ, Ω), es decir una sección $h\colon V \to A$ de $\mu\colon A \to V$. En la Sección 4.1, probamos que la sección Hamiltoniana puedes estár definida de manera equivalente por una función $F_h : A \to \mathbb{R}$ que es homogénea de grado 1 respecto del generador infinitesimal de la acción principal de $\mathbb{R}$ sobre A. Además, mostramos que una sección Hamiltoniana induce una estructura cosimpléctica sobre el espacio base V. El primer resultado de este capítulo, que motiva el estudio de este tipo de estructuras, es que las curvas integrales del correspondiente campo de Reeb son las soluciones de las ecuaciones de Hamilton definidas por h. Motivado por este resultado, definimos un *sistema Hamiltoniano no-autónomo* (A, μ, Ω, h) como un $\mathbb{R}$-fibrado principal simpléctico (A, μ, Ω) y una sección Hamiltoniana $h\colon V \to A$. Además, en la Sección 4.2, estudiamos los morfismos de estas estructuras y probamos que preservan la estructura cosimpléctica sobre el espacio base y, como consecuencia, también la dinámica del sistema. A continuación, describimos un proceso de reducción para este tipo de estructuras. Si la sección Hamiltoniana es invariante respecto de la acción canónica, entonces se puede obtener una sección Hamiltoniana reducida. Finalmente, en la Sección 4.4, estudiamos el proceso de *reconstrucción*. De hecho, describimos cómo se pueden obtener soluciones del sistema Hamiltoniano inicial a partir de soluciones de las ecuaciones de Hamilton reducidas.

El Capítulo 5 está dedicado al estudio de algunos ejemplos que pueden ser tratados usando nuestro contexto de $\mathbb{R}$-fibrados principales simplécticos. En primer lugar, supongamos que el espacio de configuración es el producto de un grupo de Lie G con la recta real, es decir $M = G \times \mathbb{R}$. En la Sección

5.1, estudiamos un proceso de reducción para el correspondiente $\mathbb{R}$-fibrado principal simpléctico estándar μ_π. Fijada una métrica Riemanniana invariante a la izquierda y una función potencial sobre G (las dos dependientes del tiempo), se obtiene un tipo particular de secciones Hamiltonianas, que se denominan *de tipo mecánico*. El grupo de simetrías es un subgrupo cerrado K de G tal que G admita una decomposición reductiva respecto de K. Entonces, obtenemos una condición suficiente para que el sistema Hamiltoniano no-autónomo reducido sea nuevamente de tipo mecánico (veáse Proposición 5.3). En las Secciones 5.2 y 5.3 discutimos dos ejemplos explícitos: el *heavy top* dependiente del tiempo ($G = SO(3)$) y el oscillador armónico amortiguado dependiente del tiempo ($G = S^1 \times \mathbb{R}$). Finalmente, en la Sección 5.4, aplicamos nuestros procesos de reducción a la formulación (independiente del sistema de referencia) de la Mecánica analítica en el espacio-tiempo Newtoniano.

Cerramos la Memoria con nuestras conclusiones y una descripción de las futuras direcciones de investigación.

Contents

Introduction

Reduction theory is an old and honored subject which began with the early roots of Mechanics, through the works of Euler, Lagrange, Jacobi, Routh, Riemann, and Poincaré. The aim of their works was "to eliminate variables" associated with symmetries in order to simplify calculations in concrete examples. The formulation of the modern reduction theory is mainly due to Marsden and Weinstein with their fundamental work on reduction of symplectic manifolds in the presence of a momentum map (see [49]). The main idea is the following one: suppose that a Lie group acts symplectically on a symplectic manifold and that a momentum map is given. A level set of the momentum map is canonically equipped with a closed 2-form which is not in general non-degenerate. Thus, one needs to quotient with respect to a suitable isotropy subgroup in order to remove such a degeneracy and to obtain an other symplectic structure.

In Differential Geometry, many applications of reduction theory are known. The first classical example is the construction of the Kirillov-Kostant-Souriau symplectic structure on the coadjoint orbit of a Lie group (see [1]). Moreover, similar ideas about reduction of geometric structures with momentum maps have been used in order to reduce not only Poisson structures ([44]), but also cosymplectic, Kähler, hyperkähler, contact, f-structures, etc. and to obtain new examples of such a kind of manifolds ([2, 17, 18, 25, 26]).

In Geometric Mechanics, most symplectic manifolds are phase spaces of momenta, i.e., cotangent bundles of a configuration space Q. If a Lie group acts on Q, then one may reduce the symplectic manifold T^*Q with respect to the corresponding cotangent lift action and a canonical momentum map. In such a case, a natural question arises: is the reduced symplectic manifold again a cotangent bundle endowed with its canonical symplectic structure? An answer to this question is given by the so-called *cotangent bundle reduction theory* (see [34, 41, 42, 53]).

Symplectic manifolds (and, in particular, cotangent bundles T^*Q of a configuration manifold Q) are the natural structures on which one may develop the Hamiltonian formulation of Classical Mechanics in the autonomous setting (see, for instance, [1, 16, 37]). In the time-dependent setting, the situation is different. The configuration space is a smooth manifold which is fibered on the real line. So, we have a fibration $\pi : M \to \mathbb{R}$, with respect to which the restricted phase space of momenta is the dual bundle $V^*\pi$ of the vertical bundle $V\pi$ of π and the extended phase space of momenta is the cotangent bundle T^*M of M (see, for instance, [14] and references therein; see also [15, 20, 27, 51]).

In this setting, a Hamiltonian function on some phase space is not sufficient to describe the dynamics of the system. Indeed, although the restricted phase space of momenta $V^*\pi$ carries a canonical Poisson structure, Hamiltonian fields for this structure are vertical with respect to the projection on time. So they cannot describe the dynamics. In the standard formulation the distinguished vector field ∂_t is added to the Hamiltonian vector field to obtain the dynamics. This can be done correctly when the fibration over time is trivial, i.e. when $M = Q \times \mathbb{R}$. When the fibration is not trivial one has to choose a reference vector field that projects onto ∂_t. Changing the reference vector field means changing the Hamiltonian. To have the description of the dynamics which is independent on the reference field one has to replace Hamiltonian functions with Hamiltonian *sections* $h : V^*\pi \to T^*M$ of the canonical projection $\mu_\pi : T^*M \to V^*\pi$. Then, using the canonical symplectic structure of T^*M (respectively, a suitable cosymplectic structure on $V^*\pi$) one may develop the extended (respectively, the restricted) Hamiltonian formalism (see Section 1.3.3 and [14, 20, 27]).

We remark that $\mu_\pi : T^*M \to V^*\pi$ is a principal $\mathbb{R}$-bundle (an *AV-bundle* in the terminology of [20], where AV stands for *Affine Values*) with principal (affine) $\mathbb{R}$-action $\psi_\pi : \mathbb{R} \times T^*M \to T^*M$ given by

$$\psi_\pi(s, \alpha_x) = \alpha_x + s(\pi^*(dt))(x), \qquad \text{for any } s \in \mathbb{R},\ \alpha_x \in T^*_xM,$$

where $\pi^*(dt)$ is the pull-back of the global 1-form dt on $\mathbb{R}$, t being the standard global coordinate on $\mathbb{R}$. In addition, the principal action is symplectic and, thus, we have the notion of a *symplectic principal* $\mathbb{R}$*-bundle*.

Generalizing these properties, we define a *non-autonomous Hamiltonian system* as a symplectic principal $\mathbb{R}$-bundle $\mu : A \to V$ and a Hamiltonian section $h : V \to A$, that is, a section of the principal $\mathbb{R}$-bundle projection μ. The Hamiltonian section h induces a vector field on V whose integral curves are the solutions of the dynamical equations for the Hamiltonian system (see Section 4.1). In the particular case when $\mu : A \to V$ is a standard symplectic principal $\mathbb{R}$-bundle (that is, $\mu = \mu_\pi$ for some fibration $\pi : M \to \mathbb{R}$), we obtain the classical Hamilton equations. Moreover, all the tools used in the standard theory in geometric non-autonomous Mechanics, as Lagrangian and

Hamiltonian formalisms or variational formulation, appear in the framework of the AV-bundles (see [20, 21, 22, 23, 60]).

We also remark that the theory of AV-bundles is closely related with the theory of Lie affgebroids which are affine analogues of Lie algebroids (for more details on these topics, see [20, 24, 27, 51, 57]).

The aim of the present thesis is to perform the reduction process in the framework of symplectic principal $\mathbb{R}$-bundles. We introduce the notion of a symmetry of a symplectic principal $\mathbb{R}$-bundle and show that, under suitable regularity conditions, one may obtain a reduced symplectic principal $\mathbb{R}$-bundle. We apply this reduction process to the standard symplectic principal $\mathbb{R}$-bundle associated with a fibration. Finally, we prove that a non-autonomous Hamiltonian system with equivariant Hamiltonian section induces a non-autonomous Hamiltonian system on the reduced symplectic principal $\mathbb{R}$-bundle.

We remark that some of the results provided in this thesis may be found in the paper [35].

Structure of the thesis: Here, we point out all the contributions in the areas of Differential Geometry and Geometric Mechanics provided by this dissertation. We also give a brief description of the contents of every chapter.

In Chapter 1, we describe briefly some basic facts about the geometric formulation of the Mechanics. In particular, we describe how symplectic and cosymplectic structures on manifolds may be applied in order to describe autonomous and non-autonomous mechanical systems. Then, we recall the main results about reduction of such a kind of structures. There are two versions for these reduction theorems which are *the point and the orbit versions*. We present these versions in Sections 1.5.1 and 1.5.2, respectively. Up to our knowledge, the bundle version for cosymplectic reduction was not precisely stated in the literature. Thus, in Section 1.5.2, we fill this gap.

If we apply symplectic reduction to the cotangent bundle of a configuration manifold Q with respect to a cotangent lift action, under suitable hypotheses, we obtain again a cotangent bundle where the canonical symplectic structure is deformed by a magnetic term. This result is described in the *embedding version* of the *cotangent bundle reduction* (see Section 1.6.2). Finally, if a principal connection is fixed, one may describe the reduced symplectic manifold in terms of the canonical symplectic structure on the reduced cotangent bundle, the Kirillov-Kostant-Souriau symplectic form on a coadjoint orbit and the curvature of the fixed connection (see *bundle version* of the cotangent bundle reduction in Section 1.6.3).

In Chapter 2, we introduce the concept of a symplectic principal $\mathbb{R}$-bundle. Firstly, we explain how the study of such a kind of structure is motivated by non-autonomous Hamiltonian systems. Then, we introduce the main object of the present thesis. We define a symplectic principal $\mathbb{R}$-bundles as a triple (A, μ, Ω), where (A, Ω) is a symplectic manifold and $\mu \colon A \to V$ is a principal $\mathbb{R}$-bundle whose principal action is symplectic. Firstly, we prove that the base manifold V of a symplectic principal $\mathbb{R}$-bundle is canonically equipped with a Poisson structure. In Section 2.2, we prove the existence of Darboux coordinates adapted to the principal fibration and we discuss morphisms of symplectic principal $\mathbb{R}$-bundles. In particular, we relate the induced Poisson structures on the corresponding base spaces of an embedding of symplectic principal $\mathbb{R}$-bundles.

An action of a Lie group G on the total space A is said to be *canonical* if the action of any element $g \in G$ on the symplectic principal $\mathbb{R}$-bundle is a morphism of the structure. Moreover, we require a suitable compatibility condition with the infinitesimal generator of the principal $\mathbb{R}$-action (see Section 2.3). Then, in Section 2.4, we describe the reduction procedure: in the presence of a canonical action and a momentum map and under regularity conditions, one may obtain a reduced symplectic principal $\mathbb{R}$-bundle. The total and the base spaces are the orbit spaces of the action of a suitable isotropy group on the level sets of the corresponding momentum maps (*point version*). Section 2.5 is devoted to the *orbit version* of the symplectic principal $\mathbb{R}$-bundle reduction. In this version, the total and the base spaces are the orbit spaces of the action of the Lie group on the preimage of a coadjoint orbit under the action of the corresponding momentum maps. We prove that the two versions are equivalent. In fact, the resultant reduced symplectic principal $\mathbb{R}$-bundles are isomorphic.

A symplectic principal $\mathbb{R}$-bundle is said to be *standard* if there exists a fibration $\pi \colon M \to \mathbb{R}$ and the principal projection is $\mu_\pi \colon T^*M \to V^*\pi$. In Chapter 3, we discuss the reduction of such symplectic principal $\mathbb{R}$-bundles. If the fibration π is invariant with respect to a free and proper action ϕ of a Lie group G on M, then, in Section 3.1, we prove that the cotangent lift of ϕ is a canonical action with momentum map on the standard symplectic principal $\mathbb{R}$-bundle $\mu_\pi \colon T^*M \to V^*\pi$. Thus, we may apply the reduction procedure. In the particular case when $\pi \colon M \to \mathbb{R}$ is a principal G-bundle, we obtain (see Section 3.2) that the base space of the reduced symplectic principal $\mathbb{R}$-bundle is a Poisson manifold of corank 1 whose symplectic leaves are isomorphic to a coadjoint orbit. On the other hand, in the general case and as in the cotangent bundle reduction theory, a natural question arise: is the reduction of a standard symplectic principal $\mathbb{R}$-bundle again standard? We discuss this problem in Section 3.3, 3.4 and 3.5. In fact, we describe two versions for the "standard reduction". In Section 3.3, we discuss the *embedding version*. We obtain, under suitable hypotheses, a new standard symplectic principal $\mathbb{R}$-bundle where the canonical symplectic 2-form on the total space is deformed by a magnetic term. In a similar way, we have a deformation of the Poisson bivector on the base space. In Section 3.4, we give a description of such a "contravariant deformation". In the *orbit version*, we obtain an other representation of the reduced structure, in which the Poisson magnetic deformation is intepreted as the vertical lift of a 2-form obtained from the curvature of a principal connection (see Section 3.5).

In Chapter 4, we introduce a dynamics in the symplectic principal $\mathbb{R}$-bundle (A, μ, Ω), which is just a section $h\colon V \to A$ of $\mu\colon A \to V$. In Section 4.1, we will prove that a Hamiltonian section of μ may be equivalently defined by a function $F_h\colon A \to \mathbb{R}$ which verifies a compatibility condition with the principal $\mathbb{R}$-action of μ (see Proposition 4.2). Moreover, a Hamiltonian section induces a cosymplectic structure on the base space V. The first result of this chapter, which motivates also the study of such a kind of structure, is that the integral curves of the corresponding Reeb vector field are just the solutions of the Hamilton equations defined by h. Thus, a *non-autonomous Hamiltonian system* (A, μ, Ω, h) is a symplectic principal $\mathbb{R}$-bundle (A, μ, Ω) and a Hamiltonian section $h\colon V \to A$. In Section 4.2, we study morphisms of the category of non-autonomous Hamiltonian systems and prove that these morphisms preserve the cosymplectic structure on the base space of the symplectic principal $\mathbb{R}$-bundle. As a consequence, they preserve the dynamics of the system. Then, in Section 4.3, we develop a reduction procedure for non-autonomous Hamiltonian systems. If the Hamiltonian section is invariant with respect to a canonical action, then a reduced Hamiltonian section is induced and the reduced dynamics is obtained from the unreduced one just by projection. Finally, in Section 4.4, we develop the so-called procedure of *reconstruction*. We suppose that a solution of the reduced Hamilton equations is known and we describe how one may obtain the motion of the unreduced non-autonomous Hamiltonian system.

Chapter 5 is devoted to the study of some examples which may be treated using our framework. Firstly, we suppose that the configuration manifold is the product of a Lie group G with the real line, i.e. $M = G \times \mathbb{R}$. Then, in Section 5.1, we develop the reduction procedure for the corresponding standard symplectic principal $\mathbb{R}$-bundle μ_π and a special kind of Hamiltonian section, induced by a time-dependent left-invariant Riemannian metric on G and by a time-dependent potential function (a *Hamiltonian section of mechanical type*). The symmetry group is a closed Lie subgroup K of G with respect to which G *admits a reductive decomposition*. We obtain also a sufficient condition for the reduced non-autonomous Hamiltonian system being again of mechanical type (see Proposition 5.3). In Sections 5.2 and 5.3, we discuss two explicit examples: the time-dependent heavy top ($G = SO(3)$) and the bidimensional time-dependent damped harmonic oscillator and the time-dependent heavy top, respectively ($G = S^1 \times \mathbb{R}$). Finally, in Section 5.4, we show how apply these reductions to the frame-independent formulation of the Analytical Mechanics in the Newtonian space-time.

The thesis ends with our conclusions and a description of future research directions.

Chapter 1

Hamiltonian (Lagrangian) Mechanics and reduction theorems

1.1 Poisson, symplectic and cosymplectic manifolds

In this first section, we recall some well-known results about Poisson, symplectic and cosymplectic manifolds (see [2, 7, 8, 61]).

A *Poisson bracket* on a manifold M is a $\mathbb{R}$-bilinear map

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

satisfying the following properties:

i) $\{\rho,\tau\} = -\{\tau,\rho\}$ (*skew-symmetry*),

ii) $\{\{\rho,\tau\},\sigma\} + \{\{\tau,\sigma\},\rho\} + \{\{\sigma,\rho\},\tau\} = 0$ (*Jacobi identity*),

iii) $\{\rho\tau,\sigma\} = \rho\{\tau,\sigma\} + \{\rho,\sigma\}\tau$ (*Leibniz rule*),

for any $\rho,\tau,\sigma \in C^\infty(M)$. Here, $C^\infty(M)$ is the space of real C^∞-functions on M.

If $\{\cdot,\cdot\}$ is a Poisson bracket on M, the couple $(M,\{\cdot,\cdot\})$ is said to be a *Poisson manifold*.

It is well-known that to give a Poisson bracket is equivalent to the existence of a bivector Λ on M such that

$$[\Lambda,\Lambda] = 0, \tag{1.1}$$

where $[\cdot,\cdot]$ is the Schouten-Nijenhuis bracket on M. The Poisson bracket $\{\cdot,\cdot\}$ and the corresponding Poisson bivector Λ are related by the following formula

$$\{\rho,\tau\} = \Lambda(\mathrm{d}\rho, \mathrm{d}\tau) \tag{1.2}$$

for any $\rho, \tau \in C^\infty(M)$.

For every Poisson manifold $(M, \{\cdot,\cdot\})$, we denote by $\sharp_\Lambda \colon T^*M \to TM$ the vector bundle morphism given by

$$\beta_x(\sharp_\Lambda(\alpha_x)) = \Lambda(x)(\alpha_x, \beta_x), \qquad \text{for any } \alpha_x, \beta_x \in T^*_xM, \tag{1.3}$$

where Λ is the corresponding Poisson bivector.

The condition *iii)* in the definition of the Poisson bracket allows to construct for each function $\tau \in C^\infty(M)$ a vector field $\mathcal{H}_\tau \in \mathfrak{X}(M)$, called the *Hamiltonian vector field associated with* τ, given by $\mathcal{H}_\tau(\rho) = \{\rho,\tau\}$. The Hamiltonian vector fields satisfy the following property

$$[\mathcal{H}_\rho, \mathcal{H}_\tau] = -\mathcal{H}_{\{\rho,\tau\}}, \qquad \text{for any } \rho, \tau \in C^\infty(M).$$

A vector field X on M is said to be *locally Hamiltonian* if, for any $x \in M$, there are an open neighbourhood U, with $x \in U$, and a function $\tau \colon U \to \mathbb{R}$ such that

$$X_{|U} = \mathcal{H}_\tau.$$

A smooth map $\varphi \colon M \to N$ between two Poisson manifolds $(M, \{\cdot,\cdot\}_M)$ and $(N, \{\cdot,\cdot\}_N)$ is said to be a *Poisson map* if

$$\{\rho,\tau\}_N \circ \varphi = \{\rho\circ\varphi, \tau\circ\varphi\}_M, \qquad \text{for any } \rho, \tau \in C^\infty(N).$$

Equivalently, if Λ_M, Λ_N are the corresponding Poisson bivectors on M and N, respectively, φ is a Poisson map if

$$(\sharp_{\Lambda_N})_{\varphi(x)} = T_x\varphi \circ (\sharp_{\Lambda_M})_x \circ T^*_x\varphi, \qquad \text{for any } x \in M, \tag{1.4}$$

where $T^*_x\varphi \colon T^*_{\varphi(x)}N \to T^*_xM$ is the dual map of the tangent map of φ at the point x.

There are many examples of Poisson manifolds (see, for instance, [61]). We are interested in symplectic manifolds, which correspond to the non-degenerate case, and in cosymplectic manifolds, which may be seen as the counterpart in odd dimension of symplectic manifolds.

A *symplectic manifold* (M, Ω) is a smooth manifold equipped with a closed non-degenerate 2-form Ω on M. There are some topological obstructions for symplectic manifolds. For instance, it's well-known that every symplectic manifold is even-dimensional and orientable. Moreover, if M is compact of dimension $2n$, its de Rham cohomology group $H^{2k}(M)$ is non-trivial, for $1 \leq k \leq n$. This implies, for example, that the only $2n$-sphere that admits a symplectic structure is the 2-sphere.

Every symplectic structure Ω on M induces a vector bundle isomorphism $\flat_\Omega \colon TM \to T^*M$ defined by

$$\flat_\Omega(X) = i_X\Omega(x), \qquad \text{with } X \in T_xM,\ x \in M.$$

Moreover, a symplectic structure Ω on M has associated a Poisson bracket $\{\cdot,\cdot\}_M$ given by

$$\{\rho,\tau\}_M = \flat_\Omega^{-1}(\mathrm{d}\tau)(\rho), \qquad \text{with } \rho,\tau \in C^\infty(M), \tag{1.5}$$

where we also have denoted by $\flat_\Omega \colon \mathfrak{X}(M) \to \Omega^1(M)$ the corresponding $C^\infty(M)$-module isomorphism. Then, the Hamiltonian vector field associated with the function $\tau \colon M \to \mathbb{R}$ is just $\mathcal{H}_\tau = \flat_\Omega^{-1}(\mathrm{d}\tau)$.

Note that a vector field X on M is locally Hamiltonian if and only if its flow preserves the symplectic form, i.e. $\mathcal{L}_X\Omega = 0$, $\mathcal{L}_X$ being the Lie derivative with respect to X.

A smooth map $\varphi \colon M \to N$ between two symplectic manifolds (M,Ω_M) and (N,Ω_N) is said to be *symplectic* if $\varphi^*\Omega_N = \Omega_M$. In general, a symplectic map φ is not a Poisson map. It is necessary to assume other conditions, for instance, that φ is a local diffeomorphism.

Remark 1.1. Let M be a Poisson manifold with Poisson bracket $\{\cdot,\cdot\}_M$. Then, we may consider the generalized distribution on M whose characteristic space at the point $x \in M$ is

$$\mathcal{F}(x) = \{\mathcal{H}_\rho(x) | \rho \in C^\infty(M)\}.$$

The distribution $\mathcal{F}$ defines a generalized foliation on M (the *symplectic foliation*) and if S is a leaf of $\mathcal{F}$, we have that the Poisson structure on M induces a symplectic structure on S. In fact, the corresponding Poisson bracket on S is given by

$$\{\rho_S,\tau_S\}_S(x) = \{\rho,\tau\}_M(x) \tag{1.6}$$

for $\rho_S,\tau_S \in C^\infty(S)$ and $x \in S$, where $\rho,\tau \in C^\infty(M)$ are arbitrary extensions of ρ_S and τ_S, respectively.

The rank (respectively, corank) of the Poisson structure at a point $x \in M$ is just the dimension (respectively, codimension) of the symplectic leaf S_x over the point x. The function $x \mapsto \dim S_x$ is lower semicontinuous, but in general is not constant. If every symplectic leaf has the same dimension, the Poisson structure is said to be *regular*. Note that a symplectic manifold is regular, because its symplectic leaves correspond just with its connected components (for more details see, for instance, [61]).

$\diamond$

A counterpart in odd dimension of symplectic manifolds are cosymplectic manifolds.

A *cosymplectic structure* on a manifold M of odd dimension $2n+1$ is a couple (ω, η), where ω is a closed 2-form on M and η is a closed 1-form on M such that $\eta \wedge \omega^n$ is a volume form on M. If (ω, η) is a cosymplectic structure on M, the triple (M, ω, η) is said to be a *cosymplectic manifold*.

Every cosymplectic structure (ω, η) on M induces a vector bundle isomorphism $\flat_{(\omega,\eta)} \colon TM \to T^*M$ given by

$$\flat_{(\omega,\eta)}(X) = i_X\omega(x) + \eta(x)(X)\eta(x), \qquad \text{with } X \in T_xM,\ x \in M.$$

Denote also by $\flat_{(\omega,\eta)}$ the corresponding $C^\infty(M)$-module isomorphism. Then the vector field $\mathcal{R} = \flat^{-1}_{(\omega,\eta)}(\eta)$ on M is called *Reeb vector field* of (M, ω, η) and it is characterized by the following conditions

$$i_{\mathcal{R}}\omega = 0, \qquad \eta(\mathcal{R}) = 1.$$

Moreover, one may induce a Poisson bracket $\{\cdot,\cdot\}_M$ on M defined by

$$\{\rho, \tau\}_M = \flat^{-1}_{(\omega,\eta)}(\mathrm{d}\tau - \mathcal{R}(\tau)\eta)(\rho), \qquad \text{with } \rho, \tau \in C^\infty(M).$$

In this case the Hamiltonian vector field associated with a function $\tau \colon M \to \mathbb{R}$ is just $\mathcal{H}_\tau = \flat^{-1}_{(\omega,\eta)}(\mathrm{d}\tau - \mathcal{R}(\tau)\eta)$. In fact, this vector field is characterized by

$$i_{\mathcal{H}_\tau}\omega = \mathrm{d}\tau - \mathcal{R}(\tau)\eta, \qquad \eta(\mathcal{H}_\tau) = 0. \tag{1.7}$$

The corresponding symplectic foliation is the completely integrable distribution of constant rank $\ker \eta$ and the symplectic form on every leaf S is just the restriction of ω to S. In particular, the Poisson structure associated with a cosymplectic manifold is regular.

Let $\varphi \colon M \to N$ be a smooth map between two cosymplectic manifolds (M, ω_M, η_M) and (N, ω_N, η_N). The map φ is said to be *cosymplectic* if φ preserves the cosymplectic structure, that is $\varphi^*\omega_N = \omega_M$ and $\varphi^*\eta_N = \eta_M$. In such a case, the Reeb vector field $\mathcal{R}_M$ of M is φ-projectable and its projection is just the Reeb vector field $\mathcal{R}_N$ of N, that is,

$$T_x\varphi(\mathcal{R}_M(x)) = \mathcal{R}_N(\varphi(x)), \qquad \text{for any } x \in M.$$

As in the symplectic case, a cosymplectic map is not, in general, a Poisson map.

1.2 Autonomous Mechanics

The use of symplectic geometry allows to obtain the classical formulation of Lagrangian and Hamiltonian Mechanics in an intrinsic way in terms of global objects as differential forms and vector fields on manifolds. The appropriate mathematical model for Lagrangian (respectively, Hamiltonian) Mechanics is the tangent bundle TQ (respectively, cotangent bundle T^*Q) of a manifold Q which represents the *configuration space* of the system. One may think elements of TQ such as velocities of curves on Q. In this sense, in Geometric Mechanics the tangent bundle TQ is also called the *phase space of velocities.*

We will present the so-called *Klein's method* in which the solutions of the Euler-Lagrange equations corresponding to a regular Lagrangian function $L\colon TQ \to \mathbb{R}$ are obtained as integral curves of a suitable vector field defined by L (see [31]). Notice that an equivalent approach is the variational one, in which the solutions of the Euler-Lagrange equations are obtained as stationary curves of the action integral (see, for instance, [16]).

From the Hamiltonian point of view, we will obtain the solutions of Hamilton equations corresponding to a Hamiltonian function $H\colon T^*Q \to \mathbb{R}$ on the cotangent bundle T^*Q of Q as integral curves of a suitable vector field on T^*Q constructed from H and the canonical symplectic structure on T^*Q (see Section 1.2.2). T^*Q is the *phase space of momenta of the system.*

The two approaches are equivalent for hyperregular Lagrangian functions (see Section 1.2.3). Indeed, in such a case, the Legendre transformation from T^*Q to TQ is a global diffeomorphism.

For more details on these topics, see [1, 16].

1.2.1 Lagrangian formulation

Let Q be a manifold, TQ its tangent bundle and $\pi_{TQ}\colon TQ \to Q$ the canonical projection. If $\gamma\colon I \to Q$ is a (smooth) curve on Q, we will denote by $\dot\gamma\colon I \to TQ$ the curve on TQ given by $t \mapsto \dot\gamma(t)$.

Recall that the Euler-Lagrange equations are second order differential equations. The correspondent global object on manifolds is given by a vector field X on TQ whose local expression is given by

$$X = \dot q^i \frac{\partial}{\partial q^i} + X^i \frac{\partial}{\partial \dot q^i}, \tag{1.8}$$

with respect to the local coordinates $(q^i, \dot q^i)$, with $i = 1, \dots, \dim Q$, induced by local coordinates (q^i) on Q. Such a kind of vector field on TQ is called a *second order differential equation* (*SODE*, for simplicity).

Note that, from (1.8), an integral curve of a SODE is necessarily a curve on TQ of type $\dot{\gamma}\colon I \to TQ$ where $\gamma\colon I \to Q$ is a curve on Q. In other words, if $t \mapsto (q^i(t))$ is the local expression of γ, then $\dot{\gamma}$ is an integral curve of X if and only if

$$\frac{d^2 q^i}{dt^2} = X^i, \qquad \text{for all } i = 1, \ldots, \dim Q.$$

This property justifies the name of a SODE for such a kind of vector fields.

An equivalent definition of a SODE involves the canonical almost tangent structure $J\colon T(TQ) \to T(TQ)$ on TQ and the Liouville vector field C on TQ. Recall that J and C are expressed locally by

$$J\left(\frac{\partial}{\partial q^i}\right) = \frac{\partial}{\partial \dot{q}^i}, \qquad J\left(\frac{\partial}{\partial \dot{q}^i}\right) = 0, \qquad C = \dot{q}^k \frac{\partial}{\partial \dot{q}^k}.$$

for $i = 1, \ldots, \dim Q$. Then, a vector field X on TQ is a SODE if and only if

$$JX = C.$$

For more details, see, for instance, [16].

If a Lagrangian function $L\colon TQ \to \mathbb{R}$ is fixed, we define the *Lagrangian energy* as the function $E_L\colon TQ \to \mathbb{R}$ given by

$$E_L = C(L) - L.$$

Moreover, we define the *Poincaré-Cartan* 1*-form associated to* L as the 1-form $\theta_L\colon TQ \to T^*(TQ)$ given by

$$\theta_L = J^*(dL),$$

where $J^*\colon T^*(TQ) \to T^*(TQ)$ is the adjoint operator of J. Then, the *Poincaré-Cartan* 2*-form* Ω_L is just (minus) the exterior derivative of θ_L, that is

$$\Omega_L = -d\theta_L.$$

The local expressions of E_L, θ_L and Ω_L in the fibred coordinates $(q^i, \dot{q}^i)$ on TQ are

$$E_L = \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L, \tag{1.9}$$

$$\theta_L = \frac{\partial L}{\partial \dot{q}^i} dq^i, \tag{1.10}$$

$$\omega_L = \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} dq^i \wedge dq^j + \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} d\dot{q}^i \wedge d\dot{q}^j. \tag{1.11}$$

From (1.11), it follows that the (closed) 2-form Ω_L is symplectic if and only if the Lagrangian function L is *regular*, that is the matrix

$$\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right)_{i,j=1,\dots,\dim Q}$$

is non-degenerate. In such a case, one may consider the Hamiltonian vector field $\mathcal{H}_{E_L}$ associated with the Lagrangian energy with respect to the symplecic form Ω_L. This vector field $\mathcal{H}_{E_L}$ is called the *Euler-Lagrange vector field associated with* L. Then, one may prove (see, for instance, [1, 16]) that $\mathcal{H}_{E_L}$ is a SODE on TQ and that the integral curves of $\mathcal{H}_{E_L}$ are just the solutions of the Euler-Lagrange equations for L. More precisely, if $\gamma\colon I \to Q$ is a curve on Q, then, $\dot{\gamma}\colon I \to TQ$ is an integral curve of $\mathcal{H}_{E_L}$ if and only if

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i} \circ \dot{\gamma}\right) - \left(\frac{\partial L}{\partial q^i} \circ \dot{\gamma}\right) = 0, \quad \text{for all } i = 1, \dots, \dim Q.$$

1.2.2 Hamiltonian formulation

In order to develop the Hamiltonian point of view, we only need to introduce a symplectic structure Ω_Q on the phase space of momenta T^*Q. The 2-form Ω_Q is defined as follows. Consider the 1-form $\theta_Q\colon T^*Q \to T^*(T^*Q)$ on T^*Q defined by

$$\theta_Q(\alpha_q)(X_{\alpha_q}) = \alpha_q(T_{\alpha_q}\pi_{T^*Q}(X_{\alpha_q})),$$

for any $\alpha_q \in T_q^*Q$ and $X_{\alpha_q} \in T_{\alpha_q}(T^*Q)$, where $\pi_{T^*Q}\colon T^*Q \to Q$ denotes the canonical projection of T^*Q on Q. The 1-form θ_Q is called the *Liouville* 1-*form on* T^*Q. It is sometimes also called the *Poincaré* 1-*form*, the *canonical* 1-*form*, or the *tautological* 1-*form*.

Then, the *canonical symplectic* 2-*form* Ω_Q *on the cotangent bundle* T^*Q is just (minus) the exterior derivative of the Liouville 1-form, i.e.

$$\Omega_Q = -d\theta_Q.$$

If (q^i), with $i = 1, \dots, \dim Q$ are local coordinates on Q, we may consider the corresponding local coordinates (q^i, p_i) on T^*Q. Then, the Liouville 1-form and the canonical symplectic form on T^*Q are respectively given by

$$\theta_Q = p_i\, dq^i, \qquad \Omega_Q = dq^i \wedge dp_i.$$

Now, let $H\colon T^*Q \to \mathbb{R}$ be a Hamiltonian fuction and consider its Hamiltonian vector field $\mathcal{H}_H \in \mathfrak{X}(T^*Q)$ with respect to Ω_Q. Then, one may easily prove that a curve on T^*Q with local expression $t \mapsto (q^i(t), p_i(t))$ is an integral curve of $\mathcal{H}_H$ if and only if it satisfies the Hamilton equations for H, i.e.

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}, \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q^i}, \qquad \text{for all } i.$$

In this sense, one may think the vector field $\mathcal{H}_H$ on T^*Q defines the *Hamiltonian dynamics* of the mechanical system.

1.2.3 The Legendre transformation

Now, we will describe the equivalence between the Lagrangian and Hamiltonian formulation for a special kind of Lagrangian functions.

Let $L \colon TQ \to \mathbb{R}$ be a Lagrangian function on the phase space of velocities. We define a smooth map $leg_L \colon TQ \to T^*Q$ as follows. For any $v_q \in T_qQ$, we denote by $leg_L(v_q) \colon T_qQ \to \mathbb{R}$ the linear map given by

$$leg_L(v_q)(w_q) = (\theta_L(v_q))(\bar{w}_q), \qquad \text{for any } w_q \in T_qQ,$$

where $\bar{w}_q \in T_{v_q}(TQ)$ is such that $T_{v_q}\pi_{TQ}(\bar{w}_q) = w_q$. Here $\theta_L : TQ \to T^*(TQ)$ denotes the Poincaré-Cartan 1-form associated to L and $\pi_{TQ} \colon TQ \to Q$ is the tangent bundle projection.

One may prove that leg_L is well-defined. yhis map is called the *Legendre transformation associated with* L. Its local expression is given by

$$leg_L(q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n) = \left(q^1, \ldots, q^n, \frac{\partial L}{\partial \dot{q}^1}, \ldots, \frac{\partial L}{\partial \dot{q}^n}\right),$$

with respect to local coordinates $(q^i, \dot{q}^i)$ on TQ and (q^i, p_i) on T^*Q induced by a coordinate system (q^i) on Q. As a consequence, the Lagrangian function L is regular if and only if the Legendre transformation $leg_L \colon TQ \to T^*Q$ is a local diffeomorphism.

The Lagrangian function L is said to be *hyperregular* if $leg_L \colon TQ \to T^*Q$ is a *global* diffeomorphism. In such a case, we may consider the Hamiltonian function $H \colon T^*Q \to \mathbb{R}$ given by

$$H = E_L \circ leg_L^{-1},$$

where $E_L \colon TQ \to \mathbb{R}$ is the Lagrangian energy associated with L.

The following results state that the Lagrangian and Hamiltonian formulations are equivalent for hyperregular Lagrangian functions.

Theorem 1.2 ([1, 16]). *Let $L\colon TQ \to \mathbb{R}$ be a hyperregular Lagrangian function on the phase space of velocities. Then, the Legendre transformation $leg_L\colon TQ \to T^*Q$ is such that*

$$leg_L^*(\theta_Q) = \theta_L,$$

where θ_Q is the Liouville 1-form on T^*Q. In particular, leg_L is a symplectic diffeomorphism between the symplectic manifolds (TQ, Ω_L) and (T^*Q, Ω_Q).

Corollary 1.3. *Let $L\colon TQ \to \mathbb{R}$ be a hyperregular Lagrangian function and $H = E_L \circ leg_L^{-1}\colon T^*Q \to \mathbb{R}$. Denote by $\mathcal{H}_{E_L} \in \mathfrak{X}(TQ)$ (respectively, $\mathcal{H}_H \in \mathfrak{X}(T^*Q)$) the Hamiltonian vector field associated with the Lagrangian function $L\colon TQ \to \mathbb{R}$ (respectively, with the Hamiltonian function $H\colon T^*Q \to \mathbb{R}$).*

Then, $\mathcal{H}_{E_L}$ and $\mathcal{H}_H$ are leg_L-related. In particular, leg_L transforms the solutions of the Euler-Lagrange equations for L into the solutions of the Hamilton equations for H.

1.3 Non-autonomous Mechanics

In the previous section, we have seen that, if a manifold Q is the configuration space of an autonomous Hamiltonian system, then the tangent bundle TQ (respectively, the cotangent bundle T^*Q) represents the phase space of velocities (respectively, the phase space of momenta) of the system. Moreover, if a regular Lagrangian function $L\colon TQ \to \mathbb{R}$ (respectively, a Hamiltonian function $H\colon T^*Q \to \mathbb{R}$) is given, one may describe the Euler-Lagrange equations (respectively, the Hamilton equations) in an intrinsic form (see, for more details, [1]).

For non-autonomous mechanical systems the situation is different (see, for instance, [14] and references therein; see also [20, 27]). Namely, *the configuration space* is a manifold M fibered over the real line. So, we have a surjective submersion $\pi\colon M \to \mathbb{R}$. In what follows, we will review briefly how Euler-Lagrange and Hamilton equations may be described in the time-dependent setting.

1.3.1 Jet bundles

Let $\pi\colon M \to \mathbb{R}$ be a surjective submersion, where M is a $(n+1)$-dimensional manifold. In other words, M is a fibered manifold over $\mathbb{R}$ with projection π. We will assume that $\mathbb{R}$ is the time, and M is the configuration space.

Denote by $J^1\pi$ the $(2n+1)$-dimensional manifold of 1-jets of local sections of π, namely

$$J^1\pi = \{j_t^1\sigma \,|\, t \in \mathbb{R},\ \sigma\colon I \subseteq \mathbb{R} \to M, \pi\circ\sigma = id_U,\ I \text{ open neighbourhood of } t\}.$$

Recall that $j_t^1\sigma$ is the equivalence class of σ with respect to the equivalence relation

$$\sigma \sim_t \sigma_1 \quad \Leftrightarrow \quad \sigma(t) = \sigma_1(t),\ \dot{\sigma}(t) = \dot{\sigma}_1(t).$$

for any local sections σ and σ_1 of π, defined in an open neighbourhood of t. Note that a local section $\sigma\colon I \to M$ of π, with $I \subseteq \mathbb{R}$ an open interval, induces a curve $j^1\sigma$ on $J^1\pi$ given by

$$j^1\sigma\colon I \to J^1\pi, \qquad t \mapsto j_t^1\sigma.$$

The curve $j^1\sigma$ is called *the* 1*-jet prolongation of* σ *to* $J^1\pi$.

Two canonical projections are given

$$\begin{aligned} \pi_{1,0}\colon J^1\pi \to M, &\qquad j_t^1\sigma \mapsto \sigma(t),\\ \pi_1\colon J^1\pi \to \mathbb{R}, &\qquad j_t^1\sigma \mapsto t. \end{aligned}$$

Recall that $J^1\pi$ is a smooth manifold of dimension $2n+1$ (see [58], for more details). Indeed, one may construct a smooth atlas on $J^1\pi$ from charts (U, t, q^i) on M adapted to the submersion π. Note that, without loss of the generality, we may suppose that the local expression of π is $(t, q^i) \mapsto t$ where the corresponding chart on $\mathbb{R}$ is the canonical one. Then, we may consider a corresponding chart on $J^1\pi$ in the form $(\pi_{1,0}^{-1}(U), t, q^i, q_1^i)$, with $i = 1, \ldots, n$. If $\sigma\colon I \to M$ is a local section of π with local expression $t \mapsto (t, \sigma^i(t))$, then $j_t^1\sigma$ has coordinates (t, q^i, q_1^i) if and only if $\sigma(t)$ has coordinates (t, q^i) and $\dot{\sigma}(t) = v_x$, where $x = \sigma(t)$ and

$$v_x = \frac{\partial}{\partial t}_{|x} + q_1^i \frac{\partial}{\partial q^i}_{|x} \tag{1.12}$$

Notice that, since σ is a section of π, the component of v_x along $\frac{\partial}{\partial t}$ is 1.

With respect to the previous coordinate system, the local expressions of $\pi_{1,0}$ and π_1 are just

$$\pi_{1,0}\colon (t, q^i, q_1^i) \mapsto (t, q^i), \qquad \pi_1\colon (t, q^i, q_1^i) \mapsto t$$

and, as a consequence, $\pi_{1,0}$ and π_1 are surjective submersions.

Note that every tangent vector $v_x \in T_xM$, with $x \in M$, has the form (1.12) if and only if $\eta(x)(v_x) = 1$, $\eta \in \Omega^1(M)$ being the (closed) 1-form $\eta = \pi^*(dt)$. Conversely, any vector $v_x \in T_xM$, with $x \in M$ and $\pi(x) = t$, is the tangent vector to a local section $\sigma\colon I \to M$ at $t \in I$ if and only if $\eta(x)(v_x) = 1$.

In other words, if we define the map

$$\iota\colon J^1\pi \to TM, \qquad j_t^1\sigma \mapsto \dot{\sigma}(t),$$

then the image of ι is just the subset of TM given by

$$\iota(J^1\pi) = \{v_x \in T_xM \mid x \in M, \eta(x)(v_x) = 1\}.$$

The local expression of ι is given by $(t, q^i, q_1^i) \mapsto (t, q^i, 1, q_1^i)$ and, as a consequence, ι is an embedding. Thus, the 1-jet bundle $J^1\pi$ may be identified with $\iota(J^1\pi)$. Note that $\iota(J^1\pi) \to M$ is an affine bundle modelled on the vector bundle $V\pi \to M$, where $V\pi$ is the vertical bundle of π.

1.3.2 The Lagrangian point of view

In what follows, we will consider a fibered manifold M over $\mathbb{R}$ with projection π. We will use the identification $J^1\pi \simeq \iota(J^1\pi)$. First of all, we will define a canonical $(1,1)$-tensor field $S\colon T(J^1\pi) \to T(J^1\pi)$ which plays the same role that the canonical almost tangent structure in the autonomous case.

In order to define S, we recall the notion of vertical lift in an affine bundle. Let $\tau_A\colon A \to M$ be an affine bundle modelled on a vector bundle $\tau\colon V \to M$. If $X \in V$, with $\tau(X) = x$, and $a_x \in A$ is in the fiber of A at x, then the vertical lift of X at a_x is the vector $X^v_{a_x} \in T_aA$ which is tangent to the curve $t \mapsto (a_x + tX)$, i.e.

$$X^v_{a_x} = \frac{d}{dt}_{|t=0}(a_x + tX).$$

Coming back to the affine bundle $J^1\pi \to M$ modelled on the vector bundle $V\pi \to M$, we define

$$S(X_{v_x}) = Y^v_{v_x},$$

for any $X_{v_x} \in T_{v_x}(J^1\pi)$, where $Y \in T_xM$ is the π-vertical tangent vector given by

$$Y = T_{v_x}\pi_{1,0}(X_{v_x}) - T_{v_x}\pi_1(X_{v_x})\, v_x.$$

With respect to the previous local coordinates (t, q^i, q_1^i) on $J^1\pi$, the local expression of S is

$$S = (dq^i - q_1^i dt) \otimes \frac{\partial}{\partial q_1^i}$$

A vector field X on $J^1\pi$ is said to be a *non-autonomous second order differential equation* (*NSODE*, for simplicity) if

$$S(X) = 0 \qquad \text{and} \qquad \eta_1(X) = 1,$$

η_1 being the 1-form on $J^1\pi$ given by $\eta_1 = \pi_1^*(dt)$. The vector field X is a NSODE if and only if it has the following local expression

$$X(t, q^i, q_1^i) = \frac{\partial}{\partial t} + q_1^i \frac{\partial}{\partial q^i} + X^i \frac{\partial}{\partial q_1^i}.$$

A local section σ of $\pi\colon M \to \mathbb{R}$ is said to be an *integral section of a NSODE* X if the 1-jet prolongation $t \mapsto j^1_t\sigma$ of σ to $J^1\pi$ is an integral curve of X. Thus, $t \mapsto (t, \sigma^i(t))$ is an integral section of X if and only if it satisfies the following system of non-autonomous differential equations of second order

$$\frac{d^2\sigma^i}{dt^2} = X^i(t, \sigma^j(t), \frac{d\sigma^j}{dt}), \qquad \frac{d\sigma^i}{dt} = q^i_A.$$

It should be remarked that an integral curve γ of a NSODE X is necessarily a 1-jet prolongation, say $\gamma = j^1\sigma$, where σ is an integral section of X (for more details, see [58]).

Now, let $L\colon J^1\pi \to \mathbb{R}$ be a Lagrangian function. Then, we will obtain the non-autonomous Euler-Lagrange equations associated with L.

We define the *Poincaré-Cartan* 1*-form associated to* L as

$$\theta_L = L\eta_1 + S^*(dL),$$

where $S^*\colon T^*(J^1\pi) \to T^*(J^1\pi)$ denotes the adjoint operator of S.

The *Poincaré-Cartan* 2*-form associated to* L is

$$\omega_L = -d\theta_L.$$

In local coordinates, we obtain

$$\begin{aligned}
\theta_L &= \frac{\partial L}{\partial q^i_1}\alpha^i + Ldt,\\
\omega_L &= \left(\frac{\partial^2 L}{\partial q^i_1 \partial t} + q^j_1 \frac{\partial^2 L}{\partial q^i_1 \partial q^j} - \frac{\partial L}{\partial q^i}\right)\alpha^i \wedge dt\\
&\quad - \frac{\partial^2 L}{\partial q^j_1 \partial q^i}\alpha^i \wedge \alpha^j + \frac{\partial^2 L}{\partial q^i_1 \partial q^j}\alpha^i \wedge dq^j_1,
\end{aligned}$$

where $\alpha^i = dq^i - q^i_1 dt$. Note also that

$$\eta_1 \wedge \omega_L^n = (-1)^{\frac{n(n+1)}{2}} n! \det\left(\frac{\partial^2 L}{\partial q^i_1 \partial q^j_1}\right) dt \wedge dq^i \wedge \cdots \wedge dq^n \wedge dq^1_1 \wedge \cdots \wedge dq^n_1.$$

As a consequence, we directly deduce the following result:

Proposition 1.4 ([14])**.** *Let $\pi\colon M \to \mathbb{R}$ be a surjective submersion and $L\colon J^1\pi \to \mathbb{R}$ be a Lagrangian function on the* 1*-jet bundle. Then, the following conditions are equivalent:*

a) The Lagrangian function $L\colon J^1\pi \to \mathbb{R}$ is regular, i.e. the matrix

$$\left(\frac{\partial^2 L}{\partial q^i_1 \partial q^j_1}\right)_{i,j=1,\dots,n}$$

is non-singular;

b) $(J^1\pi, \omega_L, \eta_1)$ *is a cosymplectic manifold.*

If condition *a)*, or equivalently condition *b)*, of the previous proposition holds, then one may prove (see, for example, [14]) that the Reeb vector field $\mathcal{R}_L$ of $(J^1\pi, \omega_L, \eta_1)$ is a NSODE. In addition, if $\gamma\colon I \to J^1\pi$ is an integral curve of $\mathcal{R}_L$ then $\gamma = j^1\sigma$, with σ a local section of π, and σ is a solution of the Euler-Lagrange equations for L, that is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_1^i} \circ j^1\sigma\right) - \frac{\partial L}{\partial q^i} \circ j^1\sigma = 0, \qquad \text{for } i = 1, \dots, n.$$

1.3.3 The Hamiltonian point of view

Let M be a $(n+1)$-dimensional fibered manifold over $\mathbb{R}$ with projection $\pi\colon M \to \mathbb{R}$.

For the Hamiltonian side, we only need to consider the vector bundle $V^*\pi \to M$ which is the dual bundle of the vertical bundle $V\pi \to M$ of π. The $(2n+1)$-dimensional manifold $V^*\pi$ is the *restricted phase space of momenta*. In this setting, the dynamics is given by a Hamiltonian *section* of a suitable fibration. In fact, we consider the cotangent bundle T^*M (the *extended phase space of momenta*) and the surjective submersion

$$\mu_\pi\colon T^*M \to V^*\pi, \qquad \alpha_x \mapsto \alpha_{x|V_x\pi}.$$

In this setting, *a Hamiltonian section* is a section $h\colon V^*\pi \to T^*M$ of μ_π. Using the Hamiltonian section, one may define a cosymplectic structure (ω_h, η) on $V^*\pi$ as follows

$$\omega_h = h^*(\Omega_M), \qquad \eta = \pi^*_{V^*\pi}(\pi^*(\mathrm{d}t)) \tag{1.13}$$

where Ω_M is the canonical symplectic 2-form on M and $\pi_{V^*\pi}\colon V^*\pi \to M$ is the corresponding vector bundle projection.

In what follows, we will give the local expressions of these objects. We fix local coordinates (t, q^i) on M such that the local expression of π is $(t, q^i) \mapsto t$, where the coordinate on $\mathbb{R}$ is the canonical one. Denote by (t, p, q^i, p_i) (respectively, (t, q^i, p_i)) the corresponding local coordinates on T^*M (respectively, on $V^*\pi$). With respect to them, the local expression of μ_π is given by

$$\mu_\pi(t, p, q^i, p_i) = (t, q^i, p_i).$$

If the Hamiltonian section $h\colon V^*\pi \to T^*M$ has local expression

$$h(t, q^i, p_i) = (t, -H(t, q, p), q^i, p_i),$$

then,

$$\omega_h = \mathrm{d}q^i \wedge \mathrm{d}p_i + \frac{\partial H}{\partial q^i}\mathrm{d}q^i \wedge \mathrm{d}t + \frac{\partial H}{\partial p_i}\mathrm{d}p_i \wedge \mathrm{d}t, \quad \eta = \mathrm{d}t.$$

Thus, the Reeb vector field $\mathcal{R}_h$ of the cosymplectic structure (ω_h, η) has the following local expression

$$\mathcal{R}_h = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}.$$

As a consequence, the integral curves of $\mathcal{R}_h$ are just the solutions of *the Hamilton equations* for h,

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}, \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q^i}, \qquad \text{for all } i.$$

For more details, see [14].

1.3.4 The Legendre transformation

In this subsection, we will construct a Legendre transformation in order to state the equivalence between the Lagrangian and the Hamiltonian approaches, for a hyperregular Lagrangian function $L\colon J^1\pi \to \mathbb{R}$.

We will suppose that $\pi\colon M \to \mathbb{R}$ is a surjective submersion and that $L\colon J^1\pi \to \mathbb{R}$ is a Lagrangian function on the 1-jet bundle of π. Again, we will identify $J^1\pi$ with $\iota(J^1\pi)$.

Consider the map $Leg_L\colon J^1\pi \to T^*M$ given by

$$(Leg_L(v_x))\,(w_x) = \theta_L(v_x)(\widetilde{w}_x),$$

where $\theta_L \in \Omega^1(J^1\pi)$ is the Poincaré-Cartan 1-form and $\widetilde{w}_x \in T_{v_x}(J^1\pi)$ is such that $T_{v_x}\pi_{1,0}(\widetilde{w}_x) = w_x$. It is clear that Leg_L is well-defined, because if we take another tangent vector $\widetilde{u}_x \in T_{v_x}(J^1\pi)$ such that $T_{v_x}\pi_{1,0}(\widetilde{u}_x) = w_x$, then $\widetilde{w}_x - \widetilde{u}_x \in \ker T_{v_x}\pi_{1,0}$ and therefore $\theta_L(v_x)(\widetilde{w}_x - \widetilde{u}_x) = 0$.

The Legendre map is the composition of Leg_L and μ_π, that is,

$$leg_L = \mu_\pi \circ Leg_L\colon J^1\pi \to V^*\pi.$$

Taking fibered coordinates (t, q^i, q^i_i), (t, p, q^i, p_i) and (t, q^i, p_i) on $J^1\pi$, T^*M and $V^*\pi$, respectively, we have that the local expression of Leg_L is

$$Leg_L(t, q^i, q^i_1) = \left(t, L - \sum_i q^i_1 \frac{\partial L}{\partial q^i_1}, q^i, \frac{\partial L}{\partial q^i_1}\right)$$

and, since $\mu_\pi(t,p,q^i,p_i) = (t,q^i,p_i)$, then

$$leg_L(t,q^i,q^i_1) = \left(t, q^i, \frac{\partial L}{\partial q^i_1}\right).$$

Consequently, we deduce another characterization for the regularity of the Lagrangian function $L\colon J^1\pi \to \mathbb{R}$.

Proposition 1.5 ([14]). *Let $\pi\colon M \to \mathbb{R}$ be a surjective submersion and $L\colon J^1\pi \to \mathbb{R}$ be a Lagrangian function on the 1-jet bundle $J^1\pi$. Then, the following statements are equivalent:*

a) The Lagrangian function $L\colon J^1\pi \to \mathbb{R}$ is regular;

b) the Legendre map $leg_L\colon J^1\pi \to V^*\pi$ is a local diffeomorphim.

In what follows, we will suppose that the Lagrangian function $L\colon J^1\pi \to \mathbb{R}$ is *hyperregular*, i.e. the corresponding Legendre transformation $leg_L\colon J^1\pi \to V^*\pi$ is a global diffeomorphism. Then, we may consider the map $h = Leg_L \circ leg_L^{-1}\colon V^*\pi \to T^*M$. Clearly, h is a Hamiltonian section of μ_π and, therefore, we may consider the cosymplectic structure (ω_h,η) on $V^*\pi$. The following result states that the Lagrangian and the Hamiltonian points of view are equivalent for hyperregular Lagrangian functions.

Theorem 1.6 ([14]). *Let $\pi\colon M \to \mathbb{R}$ be a surjective submersion and $L\colon J^1\pi \to \mathbb{R}$ be a hyperregular Lagrangian function on the 1-jet bundle $J^1\pi$. Then, the Legendre transformation $leg_L\colon J^1\pi \to V^*\pi$ is such that*

$$leg_L^*(h^*\theta_M) = \theta_L, \qquad leg_L^*\eta = \eta_1,$$

where θ_M is the Liouville 1-form on T^*M. In particular, leg_L is a cosymplectic diffeomorphism between $(J^1\pi,\omega_L,\eta_1)$ and $(V^*\pi,\omega_h,\eta)$.

Corollary 1.7. *Let $L\colon J^1\pi \to \mathbb{R}$ be a hyperregular Lagrangian function and $h = Leg_L \circ leg_L^{-1}$. Denote by $\mathcal{R}_L \in \mathfrak{X}(J^1\pi)$ (respectively, $\mathcal{R}_h \in \mathfrak{X}(V^*\pi)$) the Reeb vector field on $J^1\pi$ (respectively, on $V^*\pi$).*

Then, $\mathcal{R}_L$ and $\mathcal{R}_h$ are leg_L-related. In particular, leg_L transforms the solutions of the Euler-Lagrange equations for L into the solutions of the Hamilton equations for h.

1.4 Lie groups and left trivialization

In this Section we will recall some basic facts about Lie groups. In particular, we shall use the (left) trivialization of a Lie group G in order to describe the tangent and the cotangent bundles of G and the canonical momentum map $J\colon T^*G \to \mathfrak{g}^*$. In order to illustrate the theory, we will consider the special case when G is the Lie group $SO(3)$ (for more details, see [1]).

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and consider the action l of G on itself by left translations, that is,

$$l\colon G \times G \to G, \qquad l(g,h) = l_g(h) = gh.$$

Using the left trivialization, the tangent and the cotangent bundles TG and T^*G may be identified with the product manifolds $G \times \mathfrak{g}$ and $G \times \mathfrak{g}^*$, respectively

$$\begin{aligned} &G \times \mathfrak{g} \to TG, && (g,\xi) \in G \times \mathfrak{g} \mapsto (T_e l_g)(\xi) \in T_g G,\\ &G \times \mathfrak{g}^* \to T^*G, && (g,\nu) \in G \times \mathfrak{g}^* \mapsto (T_g^* l_{g^{-1}})(\nu) \in T_g^* G. \end{aligned}$$

Following [1], we will refer to this trivialization as defining *body coordinates.*

Under the previous identification, the symplectic structure Ω_G on T^*G is given by

$$\Omega_G(g,\nu)((\xi,\alpha),(\xi',\alpha')) = -\alpha(\xi') + \alpha'(\xi) + \nu([\xi,\xi']) \tag{1.14}$$

for $(g,\nu) \in G \times \mathfrak{g}^*$ and $(\xi,\alpha),(\xi',\alpha') \in \mathfrak{g} \times \mathfrak{g}^* \simeq T_g G \times T_\nu \mathfrak{g}^* \simeq T_{(g,\nu)}(G \times \mathfrak{g}^*)$. Using (1.14), one may give also an expression for the corresponding Poisson bracket on $T^*G \cong G \times \mathfrak{g}^*$. It is characterized by the following conditions

$$\begin{aligned} &\left\{\widehat{\xi}, \widehat{\xi'}\right\}_{T^*G} = -\widehat{[\xi,\xi']},\\ &\left\{f \circ pr_1, \widehat{\xi'}\right\}_{T^*G} = \overleftarrow{\xi'}(f) \circ pr_1,\\ &\{f \circ pr_1, f' \circ pr_1\}_{T^*G} = 0, \end{aligned} \tag{1.15}$$

for $\xi,\xi' \in \mathfrak{g}$ and $f, f' \in C^\infty(G)$, where $pr_1\colon G \times \mathfrak{g}^* \to G$ is the projection on the first factor. Here, if $\eta \in \mathfrak{g}$ then $\overleftarrow{\eta}$ is the left-invariant vector field on G whose value at the identity element of G is η and $\widehat{\eta}$ is the linear function on $G \times \mathfrak{g}^*$ defined by

$$\widehat{\eta}(g,\nu) = \nu(\eta), \qquad \text{for } (g,\nu) \in G \times \mathfrak{g}^*.$$

On the other hand, it is clear that, if K is a closed Lie subgroup of G with Lie algebra $\mathfrak{k}$, then the left translation $l\colon K \times G \to G$ is a free and proper (left) action. The infinitesimal generator ξ_G of l associated with an element $\xi \in \mathfrak{k}$ is the right-invariant vector field $\overrightarrow{\xi}$ whose value at the identity element of G is ξ. In body coordinates, it is given by

$$\xi_G(g) = \overrightarrow{\xi}(g) = (g, Ad_{g^{-1}}\xi),$$

where $Ad\colon G \times \mathfrak{g} \to \mathfrak{g}$ is the adjoint action of G. Moreover, we may consider the tangent lift action $Tl\colon K \times TG \to TG$ and the cotangent lift action $T^*l\colon K \times T^*G \to T^*G$ which are given in body coordinates respectively by

$$\begin{array}{ll} (Tl)_k(g,\xi) = (kg,\xi), & k \in K,\ (g,\xi) \in G \times \mathfrak{g} \\ (T^*l)_k(g,\nu) = (kg,\nu), & k \in K,\ (g,\nu) \in G \times \mathfrak{g}^*. \end{array} \tag{1.16}$$

On T^*G we may consider the map $J^{T^*G}\colon T^*G \to \mathfrak{k}^*$ defined by

$$J^{T^*G}(\alpha_g)(\xi) = \alpha_g(\xi_G(g)), \qquad \text{for any } \alpha_g \in T^*_gG,\ \xi \in \mathfrak{k}.$$

In other words, J^{T^*G} is given by

$$J^{T^*G}(\alpha_g) = (T^*_e r_g)(\alpha_g), \qquad \text{for } \alpha_g \in T^*_gG.$$

This map is an Ad^*-equivariant momentum map in the sense which that we will explain in the next section. In body coordinates, modulo the identification $T^*G \cong G \times \mathfrak{g}^*$, J^{T^*G} is just the composition of the coadjoint action $Coad\colon G \times \mathfrak{g}^* \to \mathfrak{g}^*$ with the map $i^*\colon \mathfrak{g}^* \to \mathfrak{k}^*$, $i\colon \mathfrak{k} \to \mathfrak{g}$ being the inclusion.

In what follows, we will describe the previous maps for the Lie group $SO(3)$ of the special orthogonal matrices under the left regular action of the subgroup K of the rotations around the z-axis given by

$$K = \left\{ \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} :\ \theta \in \mathbb{R} \right\}.$$

We will denote by A_θ the matrix of rotation of an angle θ around the z-axis. Clearly, K is diffeomorphic to the abelian Lie group S^1 via the identification $A_\theta \mapsto e^{i\theta}$.

The Lie algebra $\mathfrak{so}(3)$ of $SO(3)$ will be identified with $\mathbb{R}^3$ via the *hat map*

$$\hat{}\colon \mathbb{R}^3 \to \mathfrak{so}(3), \qquad v \mapsto \hat{v},$$

where $\hat{v}$ is the skew-symmetric matrix such that

$$\hat{v}w = v \wedge w, \qquad \text{for any } w \in \mathbb{R}^3.$$

Modulo this identification, the Lie algebra $\mathfrak{k}$ of S^1 is just $\langle e_3 \rangle \subset \mathbb{R}^3$, where e_3 is the third vector of the canonical basis of $\mathbb{R}^3$.

In body coordinates, we have the diffeomorphism

$$T^*(SO(3)) \simeq SO(3) \times (\mathbb{R}^3)^* \simeq SO(3) \times \mathbb{R}^3,$$

where we are identifying $\mathbb{R}^3$ with its dual via the canonical pairing. The cotangent lift action and the momentum map are respectively given by

$$(A_\theta, (A, \Pi)) \mapsto (A_\theta A, \Pi), \qquad (A, \Pi) \mapsto (A\Pi) \cdot e_3,$$

for any $A_\theta \in SO(3)$ and $(A, \Pi) \in SO(3) \times \mathbb{R}^3 \simeq T^*(SO(3))$.

Note that $SO(3)$ is a *reductive Lie group* (with respect to the subgroup K), i.e. the Lie algebra $\mathfrak{so}(3)$ admits the decomposition

$$\mathfrak{so}(3) \simeq \mathbb{R}^3 = \langle e_1, e_2 \rangle \oplus \mathfrak{k}$$

and $\langle e_1, e_2 \rangle$ is an $Ad(S^1)$-invariant subspace of $\mathfrak{so}(3)$ (for more details about reductive Lie groups, see, for instance, [52]). In Section 5.1, using this fact, we will construct a principal connection on the principal bundle $SO(3) \to SO(3)/S^1 \simeq S^2$. This principal connection will be important for reduction purposes.

1.5 Poisson reduction Theorems

1.5.1 Poisson, symplectic, and cosymplectic reduction

In this Section, we recall some well-known results about Poisson reduction with momentum map (for more details, see [1, 2, 42, 44, 49]).

Suppose that $\phi\colon G \times M \to M$ is an action of a Lie group G on a Poisson manifold $(M, \{\cdot, \cdot\})$. We shall denote the infinitesimal generator associated with the Lie algebra element ξ by ξ_M, i.e. ξ_M is the vector field on M such that

$$\xi_M(x) = \frac{d}{dt} \left(\phi_{\exp(tg)}(x) \right)_{|t=0}$$

and, for a real C^∞-function $\rho\colon M \to \mathbb{R}$ on M, we shall denote the Hamiltonian vector field associated with a function $\rho\colon M \to \mathbb{R}$ by $\mathcal{H}_\rho$.

A smooth map $J\colon M \to \mathfrak{g}^*$ is said to be a *momentum map* if the infinitesimal generator ξ_M associated with any $\xi \in \mathfrak{g}$ is the Hamiltonian vector field of the function $J_\xi\colon M \to \mathbb{R}$ defined by the natural pointwise pairing. In other words, if $J_\xi\colon M \to \mathbb{R}$ is the real function given by $J_\xi(x) = J(x)(\xi)$, for $x \in M$, then

$$\xi_M = \mathcal{H}_{J_\xi}. \tag{1.17}$$

Moreover, J is said to be *Ad^*-equivariant* if it is equivariant with respect to the action ϕ and to the coadjoint action $\mathrm{Ad}^*\colon G \times \mathfrak{g}^* \to \mathfrak{g}^*$, i.e.

$$J(\phi_g(x)) = \mathrm{Ad}^*_{g^{-1}}(J(x)), \qquad \text{for any } x \in M.$$

Note that the Ad^*-equivariance of a momentum map implies that the infinitesimal generator ξ_M of ϕ associated to an element $\xi \in \mathfrak{g}$ is J-related to the infinitesimal generator of the coadjoint action, i.e.

$$T_x J(\xi_M(x)) = -ad^*_\xi(J(x)), \qquad \text{for any } x \in M. \tag{1.18}$$

Here, we have used the fact that the infinitesimal generator of the coadjoint action $G \times \mathfrak{g}^* \to \mathfrak{g}^*$ of a Lie group G associated with ξ is the vector field

$$\xi_{\mathfrak{g}^*}(\nu) = -ad^*_\xi(\nu) \in \mathfrak{g}^* \simeq T_\nu \mathfrak{g}^*,$$

where $ad\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is the adjoint representation associated with the Lie algebra $\mathfrak{g}$ and $ad^*\colon \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*$ is its dual

$$ad^*_\xi \nu(\xi') = \nu(ad_\xi \xi') = \nu[\xi, \xi'], \qquad \text{for } \xi, \xi' \in \mathfrak{g},\ \nu \in \mathfrak{g}^*.$$

Condition (1.18) is called *infinitesimal Ad^*-equivariance* and it is equivalent to the fact that the map

$$J\colon \mathfrak{g} \to C^\infty(M), \qquad \xi \mapsto J_\xi$$

is a Lie algebra morphism, i.e.

$$\{J_\xi, J_{\xi'}\} = J_{[\xi,\xi']}, \qquad \text{for any } \xi, \xi' \in \mathfrak{g}.$$

Thus, an Ad^*-equivariant momentum map $J\colon M \to \mathfrak{g}^*$ is infinitesimally Ad^*-equivariant. The converse is not true. If M and G are connected, Ad^*-equivariance and infinitesimal Ad^*-equivariance are equivalent (for more details, see, for instance, [25, 45, 59]).

On the other hand, suppose that $\nu \in \mathfrak{g}^*$ is a regular value of J, so that $J^{-1}(\nu)$ is a closed submanifold of M. If G_ν denotes the isotropy group of ν with respect to the coadjoint action, i.e.

$$G_\nu = \{g \in G :\ Ad^*_g \nu = \nu\},$$

then, from the Ad^*-equivariance of J, we may induce by restriction an action

$$\phi\colon G_\nu \times J^{-1}(\nu) \to J^{-1}(\nu)$$

of G_ν on the submanifold $J^{-1}(\nu)$.

In addition, we have the following result

Theorem 1.8 (**Poisson reduction Theorem, [44]**). *Let $\phi\colon G\times M\to M$ be a free and proper Poisson action of a Lie group G on a Poisson manifold $(M,\{\cdot,\cdot\})$. If $J\colon M\to\mathfrak{g}^*$ is an Ad^*-equivariant momentum map associated with ϕ and $\nu\in\mathfrak{g}^*$ is a regular value of J, then the reduced space*

$$M_\nu = J^{-1}(\nu)/G_\nu$$

is a Poisson manifold with Poisson bracket $\{\cdot,\cdot\}_\nu$ characterized by

$$\{\rho_\nu,\tau_\nu\}_\nu(\pi_\nu(x)) = \{\rho,\tau\}(x), \qquad \text{for any } \rho_\nu,\tau_\nu\in C^\infty(M_\nu), \tag{1.19}$$

where $\pi_\nu\colon J^{-1}(\nu)\to M_\nu$ is the canonical projection and $\rho,\tau\in C^\infty(M)$ are arbitrary G-invariant extensions of $\rho_\nu\circ\pi_\nu$ and $\tau_\nu\circ\pi_\nu$, respectively.

Note that, if ρ is a G-invariant function on M and ρ_ν is the function on M_ν such that $\rho_\nu\circ\pi_\nu=\rho_{|J^{-1}(\nu)}$, then the restriction to $J^{-1}(\nu)$ of $\mathcal{H}_\rho$ is tangent to $J^{-1}(\nu)$ and

$$T_x\pi_\nu(\mathcal{H}_\rho(x)) = \mathcal{H}_{\rho_\nu}(\pi_\nu(x)), \qquad \text{for all } x\in J^{-1}(\nu). \tag{1.20}$$

The symplectic version of the Poisson reduction Theorem is the well-known Marsden-Weinstein reduction theorem.

If $\phi\colon G\times M\to M$ is a Poisson action of a Lie group G on a symplectic manifold M, then the action is *symplectic*, i.e. $\phi_g\colon M\to M$ is a symplectomorphism, for all $g\in G$. In addition, if ϕ is free then the momentum map $J\colon M\to\mathfrak{g}^*$ is a submersion (see, for example, [42], pag. 8-9) and every element of $\mathfrak{g}^*$ is a regular value of J.

Theorem 1.9 (**Marsden-Weinstein reduction Theorem, [49]**). *Suppose that $\phi\colon G\times M\to M$ is a free and proper symplectic action of a Lie group G on a symplectic manifold (M,Ω) with an associated Ad^*-equivariant momentum map $J\colon M\to\mathfrak{g}^*$. If $\nu\in\mathfrak{g}^*$, then $M_\nu=J^{-1}(\nu)/G_\nu$ is a symplectic manifold with symplectic 2-form Ω_ν characterized by*

$$\pi_\nu^*\Omega_\nu = i_\nu^*\Omega, \tag{1.21}$$

where $\pi_\nu\colon J^{-1}(\nu)\to M_\nu$ is the canonical projection and $i_\nu\colon J^{-1}(\nu)\hookrightarrow M$ is the canonical inclusion.

In fact, the Poisson structure associated with Ω_ν is just the reduced Poisson structure obtained by Theorem 1.8 (see [44]).

For cosymplectic manifolds, Albert ([2]) obtained a cosymplectic reduction Theorem which is related with the notion of a cosymplectic action.

An action $\phi\colon G\times M\to M$ of a Lie group G on a cosymplectic manifold (M,ω,η) is said to be *cosymplectic* if $\phi_g\colon M\to M$ is a cosymplectomorphism for any $g\in G$.

Theorem 1.10 (**Cosymplectic reduction Theorem, [2]**). *Let $\phi\colon G\times M\to M$ be a free, proper and cosymplectic action of a Lie group G on a cosymplectic manifold (M,ω,η). Suppose that $J\colon M\to\mathfrak{g}^*$ is an Ad^*-equivariant momentum map associated with ϕ such that*

$$\mathcal{R}(J_\xi)=0,\qquad \textit{for any } \xi\in\mathfrak{g}, \tag{1.22}$$

where $\mathcal{R}$ is the Reeb vector field of M. Then, for any $\nu\in\mathfrak{g}^*$, $M_\nu = J^{-1}(\nu)/G_\nu$ is a cosymplectic manifold with cosymplectic structure (ω_ν,η_ν) characterized by

$$\pi_\nu^*\omega_\nu = i_\nu^*\omega,\qquad \pi_\nu^*\eta_\nu = i_\nu^*\eta, \tag{1.23}$$

where $\pi_\nu\colon J^{-1}(\nu)\to M_\nu$ is the canonical projection and $i_\nu\colon J^{-1}(\nu)\hookrightarrow M$ is the canonical inclusion.

Moreover, the restriction $\mathcal{R}_{|J^{-1}(\nu)}$ of $\mathcal{R}$ to $J^{-1}(\nu)$ is tangent to $J^{-1}(\nu)$ and π_ν-projectable on the Reeb vector field $\mathcal{R}_\nu$ of M_ν.

We remark that, in the hypotheses of the previous theorem, one may prove, in a similar way as in the symplectic case, that J is a submersion (which implies that ν is a regular value of J). In fact, if $x\in M$ we must prove that the linear map $T_x^*J\colon T^*_{J(x)}\mathfrak{g}^*\simeq\mathfrak{g}\to T_x^*M$ is injective. Now, if $\xi\in\mathfrak{g}$ and $(T_x^*J)(\xi)=0$, it follows that

$$dJ_\xi(x)(v_x) = v_x(J_\xi) = 0,\qquad \text{for any } v_x\in T_xM$$

and, thus,

$$(i_{\xi_M}\omega)(x)(v_x)=0,\qquad \text{for any } v_x\in T_xM. \tag{1.24}$$

On the other hand,

$$\eta(x)(\xi_M(x)) = \eta(x)(\mathcal{H}_{J_\xi}(x)) = 0. \tag{1.25}$$

Therefore, from (1.24) and (1.25), we deduce that $\xi_M(x)=0$ and, since ϕ is free, we conclude that $\xi=0$.

In addition, from (1.7), it's easy to show that, if ρ is a G-invariant function on M and ρ_ν is the function on M_ν such that $\rho_\nu\circ\pi_\nu = \rho_{|J^{-1}(\nu)}$, then the restriction to $J^{-1}(\nu)$ of $\mathcal{H}_\rho$ is tangent to $J^{-1}(\nu)$ and

$$T_x\pi_\nu(\mathcal{H}_\rho(x)) = \mathcal{H}_{\rho_\nu}(\pi_\nu(x)),\qquad \text{for all } x\in J^{-1}(\nu). \tag{1.26}$$

This relation is formally the same of (1.20). This fact suggests that the Poisson bracket induced by the reduced cosymplectic structure is just the reduced Poisson bracket. In fact, as in the symplectic case, we have the following result

Proposition 1.11. *Under the same hypotheses as in Theorem 1.10, the Poisson bracket associated with (ω_ν, η_ν) is just the reduced Poisson bracket $\{\cdot,\cdot\}_\nu$ deduced from Theorem 1.8.*

Proof. Denote by $\{\cdot,\cdot\}_\nu$ (respectively $\{\cdot,\cdot\}'_\nu$) the Poisson bracket on M_ν obtained from Theorem 1.8 by reducing the Poisson bracket $\{\cdot,\cdot\}$ on M (respectively induced by the reduced cosymplectic structure (ω_ν, η_ν)). Let $\rho_\nu, \tau_\nu \in C^\infty(M_\nu)$ and ρ, τ be arbitrary G-invariant extensions of $\rho_\nu \circ \pi_\nu, \tau_\nu \circ \pi_\nu$ respectively. Then, for any $x \in J^{-1}(\nu)$, using (1.26) we have that

$$\begin{aligned} \{\rho_\nu, \tau_\nu\}_\nu (\pi_\nu(x)) = \{\rho, \tau\}(x) &= \mathcal{H}_\tau(x)(\rho) \\ &= \mathcal{H}_{\tau_\nu}(\pi_\nu(x))(\rho_\nu) = \{\rho_\nu, \tau_\nu\}'_\nu (\pi_\nu(x)). \end{aligned}$$

Since π_ν is surjective, $\{\cdot,\cdot\}_\nu = \{\cdot,\cdot\}'_\nu$. □

1.5.2 Orbit version

In the previous section we have presented the *point reduction*. There is another point of view that is called *orbit reduction*, which we now summarize.

First of all, we will recall the classic Kirillov-Kostant-Souriau Theorem which may be easily deduced using Marsden-Weinstein reduction Theorem (for more details, see [1]). We consider a Lie group G with Lie algebra $\mathfrak{g}$. If $\nu \in \mathfrak{g}^*$, the orbit $\mathcal{O}$ of ν with respect to the coadjoint action is diffeomorphic to the quotient manifold G/G_ν, where G_ν is the isotropy group of ν. On the other hand, since the canonical momentum map $J^{T^*G}\colon T^*G \cong G \times \mathfrak{g}^* \to \mathfrak{g}^*$ coincides with the coadjoint action, the level set $(J^{T^*G})^{-1}(\nu)$ is just given by

$$(J^{T^*G})^{-1}(\nu) = \{(g, Coad_g \nu) \,|\, g \in G\} \cong G.$$

As a consequence, using Marsden-Weinstein reduction Theorem, the coadjoint orbit

$$\mathcal{O} \cong G/G_\nu \cong (J^{T^*G})^{-1}(\nu)/G_\nu$$

is canonically equipped with a symplectic structure $\widehat{\Omega}$. The symplectic structure $\widehat{\Omega}$ is associated to various names: Lie, Borel, Weil and, more recently, Kirillov [30], Arnold [4], Kostant [33], and Souriau [59]. The 2-form $\widehat{\Omega}$ is known as the *Kirillov-Kostant-Souriau* 2-form.

Recall that an explicit expression for the symplectic 2-form $\widehat{\Omega}$ may be given in the following way. Any tangent vector at $\alpha \in \mathcal{O}$ is the infinitesimal generator of the coadjoint orbit at α, i.e. may be written in the form

$$\xi_{\mathfrak{g}^*}(\alpha) = -ad^*_\xi \alpha \in T_\alpha \mathcal{O} \subset T_\alpha \mathfrak{g}^* \cong \mathfrak{g}^*, \qquad \text{with } \xi \in \mathfrak{g}.$$

The Kirillov-Kostant-Souriau 2-form on $\mathcal{O}$ is given by

$$\widehat{\Omega}(\alpha)(-ad^*_\xi\alpha, -ad^*_{\xi'}\alpha) = -\alpha([\xi, \xi']), \tag{1.27}$$

for $\alpha \in \mathcal{O}$ and $\xi, \xi' \in \mathfrak{g}$. The sign "minus" corresponds to the left trivialization. If we consider right trivialization, we obtain another symplectic structure which differs from $\widehat{\Omega}$ for a sign.

The Poisson bracket on $\mathcal{O}$ corresponding to $\widehat{\Omega}$ is given by

$$\{\bar{\rho}_\nu, \bar{\tau}_\nu\}_{\mathcal{O}}(\alpha) = -\alpha([\mathrm{d}\bar{\rho}(\alpha), \mathrm{d}\bar{\tau}(\alpha)]), \tag{1.28}$$

for $\alpha \in \mathcal{O}$ and $\bar{\rho}_\nu, \bar{\tau}_\nu \in C^\infty(\mathcal{O})$, where $\bar{\rho}, \bar{\tau} \in C^\infty(\mathfrak{g}^*)$ are arbitrary extensions of $\bar{\rho}_\nu$ and $\bar{\tau}_\nu$, respectively, and we use the following identification $T^*_\alpha\mathfrak{g}^* \simeq (\mathfrak{g}^*)^* \simeq \mathfrak{g}$.

Now, we are able to give an other representation for the reduced Poisson structure given in Theorem 1.8. We will apply this construction to the symplectic and cosymplectic cases. We will use the level set $J^{-1}(\mathcal{O})$ of a momentum map $J\colon M \to \mathfrak{g}^*$ associated with a Poisson action $\phi\colon G \times M \to M$, where $\mathcal{O}$ is the coadjoint orbit of an element $\nu \in \mathfrak{g}^*$. Under some regularities conditions, $J^{-1}(\mathcal{O})$ is a submanifold of M as the following result states.

Lemma 1.12 (Reduction Lemma for Poisson manifolds)**.** *Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold and $\phi\colon G \times M \to M$ be a Poisson action of a Lie group G. Let $J\colon M \to \mathfrak{g}^*$ be an Ad^*-equivariant momentum map associated to this action and $\mathcal{O} \subset \mathfrak{g}^*$ the coadjoint orbit of an element $\nu \in \mathfrak{g}^*$. If ν is a regular value of J, then $J^{-1}(\mathcal{O})$ is a submanifold of M and, for any $x \in J^{-1}(\mathcal{O})$,*

$$T_x(J^{-1}(\mathcal{O})) = T_x(G \cdot x) + \ker(T_xJ). \tag{1.29}$$

Proof. The coadjoint orbit $\mathcal{O}$ is a submanifold of $\mathfrak{g}^*$. Moreover, if $x \in J^{-1}(\mathcal{O})$ then, using that J is Ad^*-equivariant, we deduce that there exists $g \in G$ such that $\phi_{g^{-1}}(x) \in J^{-1}(\nu)$. Thus,

$$x = \phi_g(x'), \qquad \text{with } x' \in J^{-1}(\nu).$$

Therefore,

$$T_xJ(T_xM) = (T_{x'}(J \circ \phi_g))(T_{x'}M)$$

which, since J is Ad^*-equivariant and ν is a regular value of J, implies that

$$(T_xJ)(T_xM) = (T_\nu Ad^*_{g^{-1}})(T_\nu\mathfrak{g}^*) = T_{J(x)}\mathfrak{g}^*.$$

Consequently, $J^{-1}(\mathcal{O})$ is a submanifold of M and

$$T_x(J^{-1}(\mathcal{O})) = (T_xJ)^{-1}(T_{J(x)}\mathcal{O}), \qquad \text{for any } x \in J^{-1}(\mathcal{O}). \tag{1.30}$$

On the other hand, by infinitesimal equivariance of J,

$$\begin{aligned} T_{J(x)}\mathcal{O} &= \{ad^*_\xi(J(x)) \,|\, \xi \in \mathfrak{g}\} = \{T_xJ(\xi_M(x)) \,|\, \xi \in \mathfrak{g}\} \\ &= \{T_xJ(v) \,|\, v \in T_x(G \cdot x)\} = (T_xJ)(T_x(G \cdot x)). \end{aligned} \tag{1.31}$$

From (1.31) and (1.30), we obtain the desired result. □

Furthermore, since the momentum map $J\colon M \to \mathfrak{g}^*$ is Ad^*-equivariant, the restricted action $G \times J^{-1}(\mathcal{O}) \to J^{-1}(\mathcal{O})$ is well-defined and we may consider the quotient space $M_{\mathcal{O}} = J^{-1}(\mathcal{O})/G$.

The next result states that if the action of G on M is free and proper then $M_{\mathcal{O}}$ is a smooth manifold equipped with a Poisson structure and $M_{\mathcal{O}}$ is diffeomorphic to the reduced Poisson manifold M_ν given in Theorem 1.8. A map between $M_\nu = J^{-1}(\nu)/G_\nu$ and $M_{\mathcal{O}} = J^{-1}(\mathcal{O})/G$ may be easily constructed using the inclusion $\iota\colon J^{-1}(\nu) \to J^{-1}(\mathcal{O})$. In fact, since ι_ν is equivariant with respect to the actions of G_ν on $J^{-1}(\nu)$ and of G on $J^{-1}(\mathcal{O})$, we may define a quotient map $[\iota]\colon M_\nu \to M_{\mathcal{O}}$. The following commutative diagram illustrates the situation

$$\begin{tikzcd} J^{-1}(\nu) \arrow[r, "\iota"] \arrow[d, "\pi_\nu"'] \arrow[rr, bend left, "i_\nu"] & J^{-1}(\mathcal{O}) \arrow[r, "i_{\mathcal{O}}"] \arrow[d, "\pi_{\mathcal{O}}"] & M \arrow[d, "\pi"] \\ M_\nu \arrow[r, "{[\iota]}"] & M_{\mathcal{O}} \arrow[r] & M/G \end{tikzcd} \tag{1.32}$$

Theorem 1.13 (**Poisson reduction Theorem, orbit version**). *Let M be a Poisson manifold with Poisson bracket $\{\cdot,\cdot\}$ and let G be a Lie group acting freely and properly on M by Poisson diffeomorphisms. Suppose that this action has an Ad^*-equivariant momentum map $J\colon M \to \mathfrak{g}^*$. Denote by $\mathcal{O}$ the orbit of an element $\nu \in \mathfrak{g}^*$ with respect to the coadjoint action.*

1) *The set $M_{\mathcal{O}} = J^{-1}(\mathcal{O})/G$ is a Poisson manifold, with Poisson bracket $\{\cdot,\cdot\}_{\mathcal{O}}$, characterized by*

$$\{\rho_{\mathcal{O}}, \tau_{\mathcal{O}}\}_{\mathcal{O}}(\pi_{\mathcal{O}}(x)) = \{\rho, \tau\}(x),$$

for any $\rho_{\mathcal{O}}, \tau_{\mathcal{O}} \in C^\infty(M_{\mathcal{O}})$. The functions $\rho, \tau \in C^\infty(M)$ are arbitrary G-invariant extensions of $\rho_{\mathcal{O}} \circ \pi_{\mathcal{O}}, \tau_{\mathcal{O}} \circ \pi_{\mathcal{O}}$, respectively, where $\pi_{\mathcal{O}}\colon J^{-1}(\mathcal{O}) \to M_{\mathcal{O}}$ is the canonical projection.

2) *If M_ν is equipped with the Poisson structure given in Theorem 1.8, the map $[\iota]\colon M_\nu \to M_{\mathcal{O}}$ is a Poisson diffeomorphism.*

We will call $M_{\mathcal{O}}$ the *orbit reduced space*. This reduced structure was first characterized by Marle [39] and by Kazhdan, Kostant, and Sternberg [29].

The orbit reduction theorem in the symplectic case is given by the following result

Theorem 1.14 (Symplectic orbit reduction, [39]). *Let $\phi\colon G\times M\to M$ be a free and proper symplectic action of a Lie group G on a symplectic manifold (M,Ω). Let $J\colon M\to\mathfrak{g}^*$ be an Ad*-equivariant momentum map associated with ϕ and $\nu\in\mathfrak{g}^*$. Then,*

1) *The quotient space $M_{\mathcal{O}}=J^{-1}(\mathcal{O})/G$ is a symplectic manifold with symplectic 2-form $\Omega_{\mathcal{O}}$ characterized by*

$$\pi_{\mathcal{O}}^*\Omega_{\mathcal{O}}=i_{\mathcal{O}}^*\Omega+J_{\mathcal{O}}^*\widehat{\Omega}, \tag{1.33}$$

where $\pi_{\mathcal{O}}\colon J^{-1}(\mathcal{O})\to M_{\mathcal{O}}$ is the canonical projection, $i_{\mathcal{O}}\colon J^{-1}(\mathcal{O})\hookrightarrow M$ is the canonical inclusion, $J_{\mathcal{O}}\colon J^{-1}(\mathcal{O})\to\mathcal{O}$ is the restriction of J to $J^{-1}(\mathcal{O})$ and $\widehat{\Omega}$ is the Kirillov-Kostant-Souriau 2-form on $\mathcal{O}$.

2) *If M_ν is equipped with the Poisson structure given in Theorem 1.9, the map $[\iota]\colon M_\nu\to M_{\mathcal{O}}$ is a symplectomorphism.*

Again, the Poisson structure associated with $\Omega_{\mathcal{O}}$ is just the Poisson structure $\{\cdot,\cdot\}_{\mathcal{O}}$ obtained by Theorem 1.13.

Now, we will describe the corresponding situation in the cosymplectic case. We will make use of the following preliminar result.

Lemma 1.15. *Let $\phi\colon G\times M\to M$ be a cosymplectic action of a Lie group G on a cosymplectic manifold (M,ω,η). Let $J\colon M\to\mathfrak{g}^*$ be an Ad*-equivariant momentum map associated with ϕ such that*

$$\mathcal{R}(J_\xi)=0,\qquad \textit{for any } \xi\in\mathfrak{g}, \tag{1.34}$$

where $\mathcal{R}\in\mathfrak{X}(M)$ is the Reeb vector field of M. Then, for any $\xi,\xi'\in\mathfrak{g}$ and $u_x\in\ker T_xJ$, with $x\in M$, we get

$$i_{\mathcal{H}_{J_\xi}}\omega=dJ_\xi, \tag{1.35}$$

$$\eta(\xi_M)=0, \tag{1.36}$$

$$\omega(\xi_M,\xi'_M)=J_{[\xi,\xi']}, \tag{1.37}$$

$$\omega(x)(\xi_M(x),u_x)=0. \tag{1.38}$$

Proof. Equations (1.35) and (1.36) directly follow from (1.7), (1.17) and (1.34). Moreover, we have that

$$\omega(\xi_M, \xi'_M) = \omega(\mathcal{H}_{J_\xi}, \mathcal{H}_{J_{\xi'}}) = dJ_\xi(\mathcal{H}_{J_{\xi'}}) = \{J_\xi, J_{\xi'}\} = J_{[\xi,\xi']},$$

where the last equality follows from the Ad^*-equivariance of J. This proves (1.37). Equation (1.38) is a consequence of (1.35). Indeed, we get

$$\omega(x)(\xi_M(x), u_x) = dJ_\xi(u_x) = (T_xJ(u_x))(\xi) = 0,$$

for any $\xi \in \mathfrak{g}$ and $u_x \in \ker T_xM$. □

Now, we may prove the orbit version of the cosymplectic reduction.

Theorem 1.16 (Cosymplectic orbit reduction). *Let $\phi\colon G \times M \to M$ be a free and proper cosymplectic action of a Lie group G on a cosymplectic manifold (M, ω, η). If $J\colon M \to \mathfrak{g}^*$ is an Ad^*-equivariant momentum map associated with ϕ such that*

$$\mathcal{R}(J_\xi) = 0, \qquad \textit{for any } \xi \in \mathfrak{g}, \tag{1.39}$$

where $\mathcal{R} \in \mathfrak{X}(M)$ is the Reeb vector field of M. Denote by $\mathcal{O}$ the coadjoint orbit of an element $\nu \in \mathfrak{g}^*$. Then,

1) *the quotient space $M_\mathcal{O} = J^{-1}(\mathcal{O})/G$ is a cosymplectic manifold with cosymplectic structure $(\omega_\mathcal{O}, \eta_\mathcal{O})$ characterized by*

$$\pi_\mathcal{O}^*\omega_\mathcal{O} = i_\mathcal{O}^*\omega + J_\mathcal{O}^*\widehat{\Omega}, \qquad \pi_\mathcal{O}^*\eta_\mathcal{O} = i_\mathcal{O}^*\eta, \tag{1.40}$$

where $\pi_\mathcal{O}\colon J^{-1}(\mathcal{O}) \to M_\mathcal{O}$ is the canonical projection, $i_\mathcal{O}\colon J^{-1}(\mathcal{O}) \to M$ is the canonical inclusion, $J_\mathcal{O}\colon J^{-1}(\mathcal{O}) \to \mathcal{O}$ is the restriction of J to $J^{-1}(\mathcal{O})$ and $\widehat{\Omega}$ is the Kirillov-Kostant-Souriau symplectic 2-form on $\mathcal{O}$.

2) *The restriction $\mathcal{R}_{|J^{-1}(\mathcal{O})}$ of $\mathcal{R}$ is tangent to $J^{-1}(\mathcal{O})$ and $\pi_\mathcal{O}$-projectable on the Reeb vector field $\mathcal{R}_\mathcal{O}$ of $M_\mathcal{O}$.*

3) *If M_ν is equipped with the cosymplectic structure given in Theorem 1.10, the map $[\iota]\colon M_\nu \to M_\mathcal{O}$ is a cosymplectic diffeomorphism.*

Proof. 1) and *3)* We must prove that the forms $i_\mathcal{O}^*\omega + J_\mathcal{O}^*\widehat{\Omega}$ and $i_\mathcal{O}^*\eta$ are basic with respect to the projection $\pi_\mathcal{O}\colon J^{-1}(\mathcal{O}) \to M_\mathcal{O} = J^{-1}(\mathcal{O})/G$. Since these forms are closed, it is sufficient to see that

$$i_{\xi_{M_{|J^{-1}(\mathcal{O})}}}(i_\mathcal{O}^*\omega + J_\mathcal{O}^*\widehat{\Omega}) = 0, \tag{1.41}$$

$$i_{\xi_{M_{|J^{-1}(\mathcal{O})}}}(i_\mathcal{O}^*\eta) = 0, \tag{1.42}$$

for $\xi \in \mathfrak{g}$ (note that the vertical bundle of $\pi_{\mathcal{O}}$ is generated by the vector fields $\xi_{M|J^{-1}(\mathcal{O})}$, with $\xi \in \mathfrak{g}$).

Now, it is clear that the map $J_{\mathcal{O}}\colon J^{-1}(\mathcal{O}) \to \mathcal{O}$ is G-equivariant. This implies that the vector field $\xi_{M|J^{-1}(\mathcal{O})}$ is $J_{\mathcal{O}}$-projectable on the vector field $\xi_{\mathfrak{g}^*|\mathcal{O}}$. Thus, from (1.17), (1.27) and (1.35), it follows that

$$\begin{aligned} i_{\xi_{M|J^{-1}(\mathcal{O})}}(i_{\mathcal{O}}^*\omega + J_{\mathcal{O}}^*\widehat{\Omega}) &= i_{\mathcal{O}}^*(dJ_\xi) + J_{\mathcal{O}}^*(i_{\xi_{\mathfrak{g}^*|\mathcal{O}}}\widehat{\Omega}) \\ &= i_{\mathcal{O}}^*(dJ_\xi) - J_{\mathcal{O}}^*(d\widehat{\xi}_{|\mathcal{O}}), \end{aligned}$$

where $\widehat{\xi}\colon \mathfrak{g}^* \to \mathbb{R}$ is the linear function on $\mathfrak{g}^*$ induced by ξ.

Therefore, using that $J_{\mathcal{O}}^*(\widehat{\xi}_{|\mathcal{O}}) = J_{\xi_{|J^{-1}(\mathcal{O})}}$, we obtain that (1.41) holds. Condition (1.42) follows from (1.36). Next, we will see that

$$[\iota]^*\omega_{\mathcal{O}} = \omega_\nu, \qquad [\iota]^*\eta_{\mathcal{O}} = \eta_\nu,$$

where (ω_ν, η_ν) is the cosymplectic structure on M_ν given in Theorem 1.10. Equivalently, we will prove that the forms $[\iota]^*\omega_{\mathcal{O}}$ and $[\iota]^*\eta_{\mathcal{O}}$ satisfy (1.23). Indeed, we have that

$$\pi_\nu^*([\iota]^*\omega_{\mathcal{O}}) = \iota^*(\pi_{\mathcal{O}}^*\omega_{\mathcal{O}}) = \iota^*(i_{\mathcal{O}}^*\Omega + J_{\mathcal{O}}^*\widehat{\Omega}) = i_\nu^*\omega,$$

where the last equality follows from the fact that $J_{\mathcal{O}} \circ \iota$ is constant and $i_{\mathcal{O}} \circ \iota = i_\nu$. Analogously, one may prove that $\pi_\nu^*([\iota]^*\eta_{\mathcal{O}}) = i_\nu^*\eta$.

2) The fact that $\mathcal{R}_{|J^{-1}(\mathcal{O})}$ is tangent to $J^{-1}(\mathcal{O})$ follows directly from (1.39). Now, let x be a point of $J^{-1}(\mathcal{O})$. Then, there exist $g \in G$ and $x' \in J^{-1}(\nu)$ such that $x = \phi_g(x')$. Moreover, since $[\iota]\colon M_\nu \to M_{\mathcal{O}}$ is a cosymplectomorphism, we get

$$\mathcal{R}_{\mathcal{O}}(\pi_{\mathcal{O}}(x)) = \mathcal{R}_{\mathcal{O}}(\pi_{\mathcal{O}}(x')) = T_{\pi_\nu(x')}[\iota](\mathcal{R}_\nu(\pi_\nu(x'))). \tag{1.43}$$

On the other hand, from Theorem 1.10, we get

$$\mathcal{R}_\nu(\pi_\nu(x')) = T_{x'}\pi_\nu(\mathcal{R}_{|J^{-1}(\nu)}(x')). \tag{1.44}$$

Comparing (1.43) and (1.44), we obtain that

$$\begin{aligned} \mathcal{R}_{\mathcal{O}}(\pi_{\mathcal{O}}(x)) &= T_{x'}([\iota] \circ \pi_\nu)(\mathcal{R}_{|J^{-1}(\nu)}(x')) \\ &= T_{x'}(\pi_{\mathcal{O}} \circ \iota)(\mathcal{R}_{|J^{-1}(\nu)}(x')) \\ &= T_{x'}\pi_{\mathcal{O}}(\mathcal{R}_{|J^{-1}(\mathcal{O})}(x')). \end{aligned}$$

Finally, since the Reeb vector field $\mathcal{R}$ is G-invariant, we conclude that

$$\mathcal{R}_{\mathcal{O}}(\pi_{\mathcal{O}}(x)) = T_x\pi_{\mathcal{O}}(\mathcal{R}_{|J^{-1}(\mathcal{O})}(x)).$$

□

1.6 Cotangent bundle reduction

1.6.1 Principal connections

In this subsection, we review briefly some well-known notions about principal connections.

We will consider the following set up. Let M be a manifold and let $\phi\colon G \times M \to M$ be a free and proper action (on the left) of a Lie group G on M. We denote by

$$\pi_{M,G}\colon M \to M/G$$

the bunble projection from M to the *shape space* M/G. We refer to $\pi_{M,G}\colon M \to M/G$ as a *principal G-bundle*. For any $x \in M$ the vertical space $V_x\pi_{M,G}$ at x is generated by the vectors $\xi_M(x)$, with ξ in the Lie algebra $\mathfrak{g}$ of G. Here $\xi_M \in \mathfrak{X}(M)$ denotes the infinitesimal generator of ϕ associated with $\xi \in \mathfrak{g}$.

Definition 1.17. *A principal connection on the principal G-bundle $\pi_{M,G}\colon M \to M/G$ is a Lie algebra valued 1-form $\lambda\colon TM \to \mathfrak{g}$ such that the following properties hold:*

i) $\lambda_x(\xi_M(x)) = \xi$ *for any* $\xi \in \mathfrak{g}$ *and* $x \in M$.

ii) λ *is equivariant with respect to the tangent action on* TM *and to the adjoint action on* $\mathfrak{g}$*, that is*

$$\lambda_{\phi_g(x)}(T_x\phi_g(X_x)) = Ad_g(\lambda_x(X_x)), \qquad \text{for } g \in G,\ x \in M, X_x \in T_xM.$$

A principal connection $\lambda\colon TM \to \mathfrak{g}$ induces a decomposition of T_xM, for all $x \in M$. Indeed, if $x \in M$ and H_x denotes the kernel of the linear map $\lambda_x\colon T_xM \to \mathfrak{g}$, then

$$T_xM = H_x \oplus V_x\pi_{M,G},$$

where $V_x\pi_{M,G}$ is the vector subspace $\ker T_x\pi_{M,G}$ of T_xM. The corresponding projections

$$\begin{aligned} \mathrm{hor}_x\colon T_xM &\to H_x \\ \mathrm{ver}_x\colon T_xM &\to V_x\pi_{M,G} \end{aligned}$$

are called *the horizontal projection* and *the vertical projection*, respectively.

On the other hand, if we fix $x \in M$, the restriction of $T_x\pi_{M,G}\colon T_xM \to T_{[x]}(M/G)$ to H_x is an isomorphism. Its inverse map will be denoted by

$$\mathrm{Hor}_x\colon T_{[x]}(M/G) \to H_x \subset T_xM$$

and is called *the horizontal lift at x*.

A direct computation shows that the following properties hold

$$\lambda_x \circ \mathrm{Hor}_x = 0, \tag{1.45}$$

$$T_x\pi_{M,G} \circ \mathrm{Hor}_x = id_{T_{[x]}(M/G)}, \tag{1.46}$$

for any $x \in M$.

Since λ is a $\mathfrak{g}$-valued 1-form, for each $\nu \in \mathfrak{g}^*$, we get a 1-form $\lambda_\nu\colon M \to T^*M$ defined by

$$\lambda_\nu(x)(X_x) = \langle \nu, \lambda_x(X_x)\rangle, \qquad \text{for } x \in M,\ X_x \in T_xM,$$

where $\langle \cdot, \cdot \rangle$ is the standard pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. This 1-form satisfies an important property with respect to the canonical momentum map $J^{T^*M}\colon T^*M \to \mathfrak{g}^*$. Recall that the momentum map $J^{T^*M}\colon T^*M \to \mathfrak{g}^*$ is defined by

$$J^{T^*M}(\alpha_x)(\xi) = \alpha_x(\xi_M(x)), \qquad \text{for } \alpha_x \in T_x^*M \text{ and } \xi \in \mathfrak{g}.$$

Proposition 1.18. *For any $\nu \in \mathfrak{g}^*$, the 1-form $\lambda_\nu\colon M \to T^*M$ takes values in $(J^{T^*M})^{-1}(\nu)$.*

For any $x \in M$, if we denote by $\lambda_x^*\colon \mathfrak{g}^* \to T_x^*M$ the dual map of the linear map $\lambda_x\colon T_xM \to \mathfrak{g}$, one may easily prove that

$$J^{T^*M} \circ T_x^*\pi_{M,G} = 0 \tag{1.47}$$

$$J^{T^*M} \circ \lambda_x^* = id_{\mathfrak{g}^*}. \tag{1.48}$$

Using (1.47) and (1.48), we deduce the following decomposition of the cotangent space T_x^*M at $x \in M$.

Proposition 1.19. *If $\lambda\colon TM \to \mathfrak{g}$ is a principal connection on the principal G-bundle $\pi_{M,G}\colon M \to M/G$, then, for any $\alpha_x \in T_x^*M$*

$$\alpha_x = T_x^*\pi_{M,G}(Hor_x^*(\alpha_x)) + \lambda_x^*(J^{T^*M}(\alpha_x)) \tag{1.49}$$

Proof. It's sufficient to compute both sides of (1.49) on vertical and horizontal vectors. □

Finally, we recall the notion of the curvature of the principal connection $\lambda\colon TM \to \mathfrak{g}$. It is a $\mathfrak{g}$-valued 2-form defined by

$$\mathrm{curv}\colon TM \times_M TM \to \mathfrak{g}, \qquad \mathrm{curv}_x(X_x, Y_x) = d\lambda(\mathrm{hor}_x(X_x), \mathrm{hor}_x(Y_x)),$$

where d is the exterior derivative. In view of the well-known Cartan formula, one may rewrite the curvature of the principal connection as

$$\mathrm{curv}(X, Y) = -\lambda([\mathrm{hor}(X), \mathrm{hor}(Y)]), \qquad \text{for } X, Y \in \mathfrak{X}(M).$$

1.6.2 Embedding version

The theory of cotangent bundle reduction is a very important special case of general reduction theory. The reduction of T^*G in Section 1.5.2 to give the symplectic structure on a coadjoint orbit is a special case of the more general procedure in which G is replaced by a configuration manifold M.

Cotangent bundle reduction theorems come in two forms, the *embedding version* and the *bundle version*, which will be treated respectively in this and in the next section.

In either case, the set up is the following one: let $\phi\colon G \times M \to M$ a free and proper action of a Lie group G on a manifold M. We will consider the cotangent lift action $T^*\phi\colon G \times T^*M \to T^*M$ given by

$$(T^*\phi)_g(\alpha_x) = (T_{\phi_g x}\phi_{g^{-1}})^*(\alpha_x), \qquad \text{for any } g \in G \text{ and } \alpha_x \in T^*_x M.$$

It's well-known (see [1]) that, if T^*M is equipped with its canonical symplectic structure Ω_M, then $T^*\phi$ is a symplectic action which is free and proper. Moreover, a canonical Ad^*-equivariant momentum map $J^{T^*M}\colon T^*M \to \mathfrak{g}^*$ is given by

$$J^{T^*M}(\alpha_x)(\xi) = J^{T^*M}_\xi(\alpha_x) = \alpha_x(\xi_M(x)) \tag{1.50}$$

for any $\alpha_x \in T^*_x M$ and $\xi \in \mathfrak{g}$. Here $\xi_M \in \mathfrak{X}(M)$ is the infinitesimal generator of ϕ associated with ξ.

Letting $\nu \in \mathfrak{g}^*$, the aim of these subsections is to determine the structure of the symplectic reduced manifold $((T^*M)_\nu, (\Omega_M)_\nu)$. We are interested in particular to describe in which way $((T^*M)_\nu, (\Omega_M)_\nu)$ is a synthesis of a cotangent bundle and a coadjoint orbit.

For the *embedding version*, we consider the so-called ν-shape space which is just the quotient space M/G_ν. Since the action of G on M is smooth, free, and proper, so is the action of the isotropy subgroup G_ν on M and, therefore, M/G_ν is a smooth manifold and the canonical projection

$$\pi_{M,G_\nu}\colon M \to M/G_\nu$$

is a surjective submersion.

Consider the action $\phi_\nu\colon G_\nu \times M \to M$ deduced from $\phi\colon G \times M \to M$. Its cotangent lift $T^*\phi_\nu\colon G_\nu \times T^*M \to T^*M$ has an Ad^*-equivariant momentum map $J^{T^*M}_\nu\colon T^*M \to \mathfrak{g}^*_\nu$ obtained by restricting J^{T^*M}, that is, for $\alpha_x \in T^*_x M$,

$$J^{T^*M}_\nu(\alpha_x) = J^{T^*M}(\alpha_x)_{|\mathfrak{g}_\nu}. \tag{1.51}$$

Let $\nu' = \nu_{|\mathfrak{g}_\nu} \in \mathfrak{g}^*_\nu$ be the restriction of ν to $\mathfrak{g}_\nu$. Since the actions are free, ν and ν' are regular values for J^{T^*M} and $J^{T^*M}_\nu$, respectively. Note that the inclusion of submanifolds

$$\bar{\iota}\colon (J^{T^*M})^{-1}(\nu) \hookrightarrow (J_\nu^{T^*M})^{-1}(\nu') \tag{1.52}$$

is a G_ν-invariant embedding.

We will use the following fact

(MT) There exists a G_ν-invariant 1-form λ_ν on M with values in $(J_\nu^{T^*M})^{-1}(\nu')$.

Notice that, if $\lambda\colon TM \to \mathfrak{g}$ is the connection 1-form associated with a principal connection on the principal G-bundle $\pi_{M,G}\colon M \to M/G$ then $\lambda_\nu = \nu \circ \lambda$ defines a 1-form on M which satisfies the condition *(MT)* (see Proposition 1.18; for more details, see [42]).

Now, using that the 2-form $\mathrm{d}\lambda_\nu$ is basic with respect to the projection π_{M,G_ν}, we deduce that there exists a unique closed 2-form β_{λ_ν} on M/G_ν such that

$$\pi^*_{M,G_\nu}\beta_{\lambda_\nu} = \mathrm{d}\lambda_\nu. \tag{1.53}$$

Then, we define the 2-form B_{λ_ν} on $T^*(M/G_\nu)$ as

$$B_{\lambda_\nu} = \pi^*_{M/G_\nu}\beta_{\lambda_\nu},$$

where $\pi_{M/G_\nu}\colon T^*(M/G_\nu) \to M/G_\nu$ is the cotangent bundle projection. The form B_{λ_ν} is usually called *the magnetic term associated with* λ_ν (see, for instance, [42]).

The embedding version of the cotangent bundle reduction theorem states that the reduced symplectic manifold $(T^*M)_\nu$ can be embedded in the cotangent manifold $T^*(M/G_\nu)$ where the canonical symplectic 2-form Ω_{M/G_ν} on $T^*(M/G_\nu)$ is deformed by the magnetic term B_{λ_ν}. More precisely, we have that

Theorem 1.20. *Let $\pi_{M,G}\colon M \to M/G$ be a principal bundle with (free and proper) principal action $\phi\colon G \times M \to M$ and $\nu \in \mathfrak{g}^*$. Choose a G_ν-invariant 1-form $\lambda_\nu \in \Omega^1(M)$ with values in $(J_\nu^{T^*M})^{-1}(\nu')$. Then there is a symplectic embedding*

$$\varphi_{\lambda_\nu}\colon ((T^*M)_\nu, (\Omega_M)_\nu) \to (T^*(M/G_\nu), \Omega_{M/G_\nu} - B_{\lambda_\nu}),$$

between the reduced symplectic manifold $((T^*M)_\nu, (\Omega_M)_\nu)$ and the symplectic manifold $(T^*(M/G_\nu), \Omega_{M/G_\nu} - B_{\lambda_\nu})$, where the canonical symplectic structure Ω_{M/G_ν} on the cotangent bundle $T^*(M/G_\nu)$ is deformed by $B_{\lambda_\nu} \in \Omega^2(T^*(M/G_\nu))$, the magnetic term associated with λ_ν.

Moreover, φ_{λ_ν} is a symplectomorphism if and only if $\mathfrak{g} = \mathfrak{g}_\nu$ (in particular, if $G = G_\nu$), where $\mathfrak{g}_\nu$ is the Lie algebra of G_ν.

An explicit expression of the symplectic embedding φ_{λ_ν} may be given (for more details, see, for instance, [1, 42]). Consider the map

$$\bar{\varphi}_{\lambda_\nu} \colon (J_\nu^{T^*M})^{-1}(\nu') \to T^*(M/G_\nu)$$

given by

$$\bar{\varphi}_{\lambda_\nu}(\alpha_x)(T_x\pi_{M,G_\nu}(v_x)) = (\alpha_x - \lambda_\nu(x))(v_x)$$

for all $\alpha_x \in (J_\nu^{T^*M})^{-1}(\nu') \cap T_x^*M$ and $v_x \in T_xM$. This map is invariant with respect to

$$\phi \colon G_\nu \times (J_\nu^{T^*M})^{-1}(\nu') \to (J_\nu^{T^*M})^{-1}(\nu').$$

We will denote by $\widetilde{\varphi}_{\lambda_\nu}$ the corresponding quotient map from $(J_\nu^{T^*M})^{-1}(\nu')/G_\nu$ to $T^*(M/G_\nu)$. One may prove (see, for instance, [42]) that $\widetilde{\varphi}_{\lambda_\nu}$ is a diffeomorphism.

On the other hand, since the embedding $\bar{\iota}$ given in (1.52) is G_ν-equivariant, we get a smooth map

$$\iota \colon (J^{T^*M})^{-1}(\nu)/G_\nu \hookrightarrow (J_\nu^{T^*M})^{-1}(\nu')/G_\nu. \tag{1.54}$$

Then, the required embedding φ_{λ_ν} is just the composition of ι with $\widetilde{\varphi}_{\lambda_\nu}$. The expression of φ_{λ_ν} will be useful in the sequel.

1.6.3 Bundle version

For the *bundle version* of the cotangent reduction theory, we will fix a principal connection on the principal bundle $\pi_{M,G} \colon M \to M/G$. Let $\lambda \colon TM \to \mathfrak{g}$ be the corresponding connection 1-form and denote by $\phi \colon G \times M \to M$ the principal action.

As we already know, the cotangent lift action $T^*\phi \colon G \times T^*M \to T^*M$ is symplectic and admits an Ad^*-equivariant momentum map $J^{T^*M} \colon T^*M \to \mathfrak{g}^*$ given by (1.50). If we apply the orbit version of the reduction procedure to the symplectic manifold (T^*M, Ω_M) (see Section 1.5.2), we obtain that the reduced symplectic manifold is $((T^*M)_{\mathcal{O}}, (\Omega_M)_{\mathcal{O}})$, $\mathcal{O}$ being the coadjoint orbit of an element $\nu \in \mathfrak{g}^*$ and $(T^*M)_{\mathcal{O}} = (J^{T^*M})^{-1}(\mathcal{O})/G$.

On the other hand, one may give an other description of the reduced symplectic manifold $((T^*M)_{\mathcal{O}}, (\Omega_M)_{\mathcal{O}})$ as a synthesis of a cotangent bundle and a suitable vector bundle constructed with the coadjoint orbit $\mathcal{O}$ (for more details, see [43]). Precisely, we consider the diagonal action of the Lie group G on the product manifold $M \times \mathcal{O}$ given by

$$G \times (M \times \mathcal{O}) \to M \times \mathcal{O} \qquad (g, (x, \kappa)) \mapsto (\phi_g x, Coad_g \kappa).$$

Since this action is free and proper, we may consider the projection $p_G\colon M\times\mathcal{O}\to\widetilde{\mathcal{O}}$, where $\widetilde{\mathcal{O}}$ denotes the quotient space $\widetilde{\mathcal{O}}=(M\times\mathcal{O})/G$. Moreover, we consider the fibration

$$T^*(M/G)\times_{M/G}\widetilde{\mathcal{O}}\to M/G.$$

Now, we will equip the total space $T^*(M/G)\times_{M/G}\widetilde{\mathcal{O}}$ with a symplectic 2-form Ω_{red}. In order to define the 2-form Ω_{red}, we need to introduce some preliminary objects.

We consider the 1-form α_λ on $M\times\mathcal{O}$ defined by

$$\alpha_\lambda(x,\kappa)(v_x,X_\kappa)=\langle\kappa,\lambda_x(v_x)\rangle,$$

for any $(x,\kappa)\in M\times\mathcal{O}$ and $(v_x,X_\kappa)\in T_{(x,\kappa)}(M\times\mathcal{O})\simeq T_xM\times T_\kappa\mathcal{O}$. Here, $\lambda\colon TM\to\mathfrak{g}$ is the fixed connection 1-form.

One may give also an explicit expression for the differential of the 1-form α_λ. Let $(v_x,-ad^*_\kappa\xi'),(w_x,-ad^*_\kappa\eta')\in T_xM\times T_\kappa\mathcal{O}\simeq T_{(x,\kappa)}(M\times\mathcal{O})$. Let

$$\xi=\lambda_x(v_x)\qquad\text{and}\qquad\eta=\lambda_x(w_x)$$

so that

$$v_x=\xi_M(x)+\mathrm{hor}_x v_x,\qquad w_x=\eta_M(x)+\mathrm{hor}_x w_x,$$

where $\mathrm{hor}_x\colon T_xM\to H_x$ denotes the horizontal projection corresponding to the connection λ. Then, we have (see [43])

$$\begin{aligned}&d\alpha_\lambda(x,\kappa)\big((v_x,-ad^*_{\xi'}\kappa),(w_x,-ad^*_{\eta'}\kappa)\big)\\&\qquad=\langle\kappa,[\eta',\xi]\rangle+\langle\kappa,[\eta,\xi']\rangle+\langle\kappa,[\xi,\eta]\rangle+\langle\kappa,\mathrm{curv}(v_x,w_x)\rangle,\end{aligned}\tag{1.55}$$

where $[\cdot,\cdot]\colon\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ is the Lie bracket on $\mathfrak{g}$ and curv is the curvature $\mathfrak{g}$-valued 2-form corresponding to λ.

Moreover, one may prove that there exists a unique 2-form $\bar\beta_\lambda$ on $\widetilde{\mathcal{O}}$ such that

$$p_G^*\bar\beta_\lambda=d\alpha_\lambda-pr_2^*\widehat{\Omega},\tag{1.56}$$

where $pr_2\colon M\times\mathcal{O}\to\mathcal{O}$ is the second projection and $\widehat{\Omega}$ is the Kirillov-Kostant-Souriau symplectic form on $\mathcal{O}$.

Now, we may state the orbit bundle version of the cotangent bundle theorem.

Theorem 1.21 ([43]). *Let $\pi_{M,G}\colon M\to M/G$ be a principal bundle such that the principal action $\phi\colon G\times M\to M$ is free and proper. Consider the symplectic manifold $((T^*M)_{\mathcal{O}},(\Omega_M)_{\mathcal{O}})$ obtained by reducing the cotangent bundle (T^*M,Ω_M) with respect to the cotangent lift action $T^*\phi$.*

If we fix a principal connection $\lambda \colon TM \to \mathfrak{g}$ *on the principal bundle* $\pi_{M,G} \colon M \to M/G$*, then the couple*

$$(T^*(M/G) \times_{M/G} \widetilde{\mathcal{O}}, \Omega_{red})$$

is a symplectic manifold, where Ω_{red} *is the* 2*-form defined by*

$$\Omega_{red} = \Omega_{M/G} - \bar{\beta}_\lambda,$$

and $\Omega_{M/G}$ *is the canonical symplectic* 2*-form on* $T^*(M/G)$*.*

Moreover, the map $\Theta \colon (T^*M)_{\mathcal{O}} = (J^{T^*M})^{-1}(\mathcal{O})/G \to T^*(M/G) \times_{M/G} \widetilde{\mathcal{O}}$ *given by*

$$\pi_{\mathcal{O}}(\alpha_x) \mapsto (Hor_x^* \alpha_x, [x, J^{T^*M}(\alpha_x)])$$

is a symplectomorphism, that is

$$\Theta^* \Omega_{red} = (\Omega_M)_{\mathcal{O}}.$$

Chapter 2

Reduction of symplectic principal $\mathbb{R}$-bundles

2.1 Symplectic principal $\mathbb{R}$-bundles: a motivation

In Section 1.3, we have shown that non-autonomous Hamiltonian systems, where the configuration space is a fibered manifold $\pi : M \to \mathbb{R}$ over the real line, may be described using the jet bundle theory in the Lagrangian picture and a suitable projection μ_π in the Hamiltonian side. Moreover, for hyperregular Lagrangian functions the two approaches are equivalent.

From now on, we will follow the Hamiltonian point of view. First of all, we remark that the projection $\mu_\pi : T^*M \to V^*\pi$ is the projection of a principal $\mathbb{R}$-bundle. Precisely, a principal action $\psi_\pi : \mathbb{R} \times T^*M \to T^*M$ is given. It is defined by

$$\psi_\pi(s, \alpha_x) = \alpha_x + s(\pi^*(dt))(x), \qquad \text{for } s \in \mathbb{R} \text{ and } \alpha_x \in T^*_xM.$$

Here, dt is the (global) 1-form on $\mathbb{R}$ obtained from the standard global coordinate t on $\mathbb{R}$ and $\pi^*(dt)$ is its pull-back to M.

The action ψ_π is free and the quotient space $T^*M/\mathbb{R}$ may be canonically identified with $V^*\pi$.

The manifold $V^*\pi$ admits a linear Poisson structure which is characterized as follows. If $\Gamma(V\pi)$ is the space of sections of the vector bundle $\pi_{V\pi} : V\pi \to M$ and $X \in \Gamma(V\pi)$, then X induces a linear function $\widehat{X}^{V^*\pi} : V^*\pi \to \mathbb{R}$ on the restricted phase space of momenta given by

$$\widehat{X}^{V^*\pi}(\alpha_x) = \alpha_x(X(x)), \qquad \text{for any } \alpha_x \in V^*_x\pi, \text{ with } x \in M.$$

On the other hand, if $f \in C^\infty(M)$ then $f \circ \pi_{V^*\pi} : V^*\pi \to \mathbb{R}$ is a basic function on $V^*\pi$. Here, $\pi_{V^*\pi} : V^*\pi \to M$ is the vector bundle projection of the dual bundle of $V\pi$. The Poisson bracket $\{\cdot,\cdot\}_{V^*\pi}$ on $V^*\pi$ is characterized by the following relations

$$\begin{aligned} \left\{\widehat{X}^{V^*\pi}, \widehat{Y}^{V^*\pi}\right\}_{V^*\pi} &= -\widehat{[X,Y]}^{V^*\pi}, \\ \left\{f \circ \pi_{V^*\pi}, \widehat{Y}^{V^*\pi}\right\}_{V^*\pi} &= Y(f) \circ \pi_{V^*\pi}, \\ \{f \circ \pi_{V^*\pi}, f' \circ \pi_{V^*\pi}\}_{V^*\pi} &= 0, \end{aligned} \tag{2.1}$$

for $X, Y \in \Gamma(V\pi)$ and $f, f' \in C^\infty(M)$. Note that if $X, Y \in \Gamma(V\pi)$, then the Lie bracket $[X, Y]$ also belongs to $\Gamma(V\pi)$.

The extended phase space of momenta T^*M also admits a linear Poisson structure. In fact, the Poisson bracket $\{\cdot,\cdot\}_{T^*M}$ on T^*M is induced by the canonical symplectic structure Ω_M on T^*M. In addition, the canonical projection $\mu_\pi : T^*M \to V^*\pi$ is a Poisson epimorphism.

If (t, q^i) are local coordinates on M adapted to the submersion π (t being the standard coordinate on $\mathbb{R}$), then we may consider the corresponding local coordinates (t, p, q^i, p_i) on T^*M and (t, q^i, p_i) on $V^*\pi$. Moreover, we have that

$$\left\{t, q^i\right\}_{V^*\pi} = \{t, p_i\}_{V^*\pi} = \left\{q^i, q^j\right\}_{V^*\pi} = \{p_i, p_j\}_{V^*\pi} = 0, \quad \left\{q^i, p_j\right\}_{V^*\pi} = \delta^i_j.$$

As we have already seen (see Subsection 1.3.3), a Hamiltonian section $h : V^*\pi \to T^*M$ of μ_π induces a cosymplectic structure (ω_h, η) on $V^*\pi$, in such a way the corresponding Reeb vector field $\mathcal{R}_h$ describes the dynamics of the system.

We also note that, if a Hamiltonian section $h : V \to A$ is fixed, for any $\alpha_x \in T^*_x M$, with $x \in M$, the elements $(h \circ \mu_\pi)(\alpha_x)$ and α_x are in the same orbit of the action ψ_π, that is

$$\mu_\pi((h \circ \mu_\pi)(\alpha_x)) = \mu_\pi(\alpha_x).$$

Thus, there exists a unique $F_h(\alpha_x) \in \mathbb{R}$ such that

$$\psi_\pi(-F_h(\alpha_x), \alpha_x) = h(\mu_\pi(\alpha_x)).$$

The *extended Hamiltonian function* associated with the Hamiltonian section h is just the real C^∞-function $F_h : T^*M \to \mathbb{R}$.

It's easy to prove that the Hamiltonian vector field $\mathcal{H}_{F_h} \in \mathfrak{X}(T^*M)$ of F_h is μ_π-projectable on the Reeb vector field $\mathcal{R}_h$ of the cosymplectic structure (ω_h, η). Indeed, the local expression of the principal action ψ_π is

$$\psi_\pi(s, (t, p, q^i, p_i)) = (t, s + p, q^i, p_i).$$

Therefore,

$$F_h(t,p,q^i,p_i) = p + H(t,q^i,p_i)$$

and

$$\mathcal{H}_{F_h} = \frac{\partial}{\partial t} - \frac{\partial H}{\partial t}\frac{\partial}{\partial p} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}.$$

Consequently, the solutions of the Hamilton equations for the Hamiltonian section h are just the projection (via μ_π) of the solutions of the Hamilton equations for the extended Hamiltonian function F_h.

Remark 2.1. Note that the principal action ψ_π is symplectic. In fact, the infinitesimal generator Z_{μ_π} of ψ_π is the Hamiltonian vector field of the real function on T^*M given by

$$-\pi \circ \pi_M : T^*M \to \mathbb{R}$$

where $\pi_M : T^*M \to M$ is the canonical projection. This motivates our definition of *symplectic principal* $\mathbb{R}$*-bundle* as a principal $\mathbb{R}$-bundle whose total space is symplectic and whose principal action is symplectic (see Section 2.2). We will see that this definition allows to construct a framework which describes non-autonomous Hamiltonian systems.

◇

Remark 2.2. Under the previous hypotheses, the 1-form η given by (1.13) may be expressed as

$$\eta = h^*(i_{\mathcal{H}_{\pi\circ\pi_M}}\Omega_M)) = -h^*(i_{Z_{\mu_\pi}}\Omega_M).$$

Thus, the cosymplectic structure may be constructed only from the symplectic principal $\mathbb{R}$-bundle and a Hamiltonian section $h : V^*\pi \to T^*M$. This motivates our definition of non-autonomous Hamiltonian system in Section 4.1.

◇

2.2 The category of symplectic principal $\mathbb{R}$-bundles

Motivated by the example of the above section, one may introduce the notion of a symplectic principal $\mathbb{R}$-bundle as follows.

Let $\mu : A \to V$ be a principal $\mathbb{R}$-bundle (an *AV-bundle* in the terminology of [20]). We will denote by

$$\psi : \mathbb{R} \times A \to A, \qquad (s,a) \mapsto \psi_s(a), \tag{2.2}$$

the corresponding principal action of the Lie group $(\mathbb{R},+)$ on the manifold A.

In this case the vertical distribution of μ has dimension 1 and it is generated by the infinitesimal generator $Z_\mu \in \mathfrak{X}(A)$ whose flow is ψ.

Definition 2.3. *We will say that $\mu : (A,\Omega) \to V$ is a* symplectic principal $\mathbb{R}$-bundle*, if $\mu : A \to V$ is a principal $\mathbb{R}$-bundle and Ω is a symplectic structure on A such that the associated principal action $\psi : \mathbb{R} \times A \to A$ is symplectic.*

Note that the infinitesimal generator Z_μ of a symplectic principal $\mathbb{R}$-bundle $\mu : (A,\Omega) \to V$ is a locally Hamiltonian vector field, i.e.

$$\mathcal{L}_{Z_\mu}\Omega = 0.$$

Remark 2.4. If $\pi : M \to \mathbb{R}$ is a surjective submersion then T^*M is the total space of a symplectic principal $\mathbb{R}$-bundle over $V^*\pi$ (see Section 2.1). It is the *standard symplectic principal $\mathbb{R}$-bundle associated with the fibration* $\pi : M \to \mathbb{R}$.

$\diamond$

Remark 2.5. The standard symplectic principal $\mathbb{R}$-bundle associated with a fibration and a magnetic term. Let $\pi : M \to \mathbb{R}$ be a surjective submersion with total space a manifold M of dimension $n+1$ and β a closed 2-form on M. Consider the closed basic 2-form (*the magnetic term*) $B = \pi_M^*\beta$ on T^*M, where $\pi_M : T^*M \to M$ is the canonical projection. An easy computation shows that B is invariant with respect to the $\mathbb{R}$-principal action of the standard symplectic principal $\mathbb{R}$-bundle $\mu_\pi : T^*M \to V^*\pi$. Thus, if Ω_M is the canonical symplectic form on T^*M, $\mu_\pi : (T^*M, \Omega_M - B) \to V^*\pi$ is a symplectic principal $\mathbb{R}$-bundle.

$\diamond$

If $\mu : (A,\Omega) \to V$ is a symplectic principal $\mathbb{R}$-bundle, then, using a well-known result on Poisson reduction (see, for instance, [53], Theorem 10.1.1) we have that the base manifold V may be canonically equipped with a Poisson structure as we show in the following result.

Proposition 2.6. *Let $\mu : (A,\Omega) \to V$ be a symplectic principal $\mathbb{R}$-bundle. Then, there exists a unique Poisson structure $\{\cdot,\cdot\}_V$ on V such that μ is a Poisson map, i.e.*

$$\{f\circ\mu, f'\circ\mu\}_A = \{f,f'\}_V \circ \mu, \qquad \textit{for any } f,f' \in C^\infty(V), \tag{2.3}$$

where $\{\cdot,\cdot\}_A$ is the Poisson bracket on A induced by the symplectic form Ω.

Now, we will prove a version of Darboux Theorem for a symplectic principal $\mathbb{R}$-bundle.

Theorem 2.7. *Let $\mu : (A, \Omega) \to V$ be a symplectic principal $\mathbb{R}$-bundle with infinitesimal generator Z_μ. Suppose that* $\dim A = 2n + 2$*. Then, for any $a \in A$, there exist local coordinates (t, p, q^i, p_i), $(i = 1, \ldots, n)$ in a neighborhood U of a such that*

i) the local expression of $\mu : A \to V$ is

$$\mu(t, p, q^i, p_i) = (t, q^i, p_i), \tag{2.4}$$

ii) (t, p, q^i, p_i) are Darboux coordinates for Ω.

Moreover, the local expression of the infinitesimal generator is $Z_\mu = \frac{\partial}{\partial p}$.

Proof. The proof is based on the well-known construction of the Darboux coordinates (see, for instance, [5, 37]).

Fix $a \in A$. Since the vector field Z_μ is locally Hamiltonian, there exists a function t defined in an open neighborhood of a such that $Z_\mu = -\mathcal{H}_t$. Choose a function p (eventually defined on a smaller open neighborhood of a) such that $Z_\mu(p) = 1$. Note that $\mathcal{H}_p, \mathcal{H}_t$ are linearly independent. Indeed, we have that

$$\Omega(\mathcal{H}_p, \mathcal{H}_t) = \mathrm{d}p(\mathcal{H}_t) = -Z_\mu(p) = -1.$$

Define the closed 2-form $\Omega' = \Omega - \mathrm{d}t \wedge \mathrm{d}p$. Since $i_{\mathcal{H}_t}\Omega' = i_{\mathcal{H}_p}\Omega' = 0$, the rank of Ω' is $2n$. From generalized Darboux Theorem, there exist coordinates $(\bar{t}, \bar{p}, q^i, p_i)$ on an open neighborhood of a such that $\Omega' = \sum_i \mathrm{d}q^i \wedge \mathrm{d}p_i$. Thus,

$$\Omega = \mathrm{d}t \wedge \mathrm{d}p + \sum_i \mathrm{d}q^i \wedge \mathrm{d}p_i.$$

Since Ω^{n+1} is a volume form on A, (t, p, q^i, p_i) are coordinates on A and obviously are Darboux coordinates with respect to Ω. Note that

$$Z_\mu(t) = -\mathcal{H}_t(t) = 0, \qquad Z_\mu(p) = 1,$$
$$Z_\mu(q^i) = -\mathcal{H}_{q^i}(t) = \frac{\partial}{\partial p_i}(t) = 0, \qquad Z_\mu(p_i) = -\mathcal{H}_{p_i}(t) = -\frac{\partial}{\partial q^i}(t) = 0.$$

Thus, $Z_\mu = \frac{\partial}{\partial p}$. Since μ is locally the projection of A with coordinates (t, p, q^i, p_i) on $A/\langle Z_\mu \rangle$, the local expression of μ is as in (2.4). $\square$

We will say that (t, p, q^i, p_i) in the previous theorem are *canonical coordinates* for the symplectic principal $\mathbb{R}$-bundle μ.

Note that in such a case (t, q^i, p_i) are coordinates on V and the corresponding local expression of the induced Poisson bracket on V with respect to these coordinates is the following one:

$$\{t, q^i\}_V = \{t, p_i\}_V = \{q^i, q^j\}_V = \{p_i, p_j\}_V = 0, \qquad \{q^i, p_j\}_V = \delta^i_j.$$

In the last part of this section we will study morphisms in the category of symplectic principal $\mathbb{R}$-bundles.

Let $\mu : A \to V$ and $\mu' : A' \to V'$ be two principal $\mathbb{R}$-bundles with principal actions ψ and ψ', respectively. Suppose that the function $\varphi : A \to A'$ is a principal $\mathbb{R}$-bundle morphism, that is, φ is equivariant with respect to the principal actions, i.e.

$$\varphi \circ \psi_s = \psi'_s \circ \varphi, \qquad \text{for any } s \in \mathbb{R}. \tag{2.5}$$

From (2.5), one deduces that the infinitesimal generators Z_μ and $Z_{\mu'}$ of $\mu : A \to V$ and $\mu' : A' \to V'$ respectively, are φ-related, i.e.

$$(T_a\varphi)(Z_\mu(a)) = Z_{\mu'}(\varphi(a)), \qquad \text{for all } a \in A. \tag{2.6}$$

Moreover, by passing to the quotient and using (2.5), one may define a map $\varphi^V : V \to V'$ characterized by the following relation

$$\mu' \circ \varphi = \varphi^V \circ \mu. \tag{2.7}$$

Note that, since μ is a submersion, then φ^V is smooth. The following diagram illustrates the situation

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\mu"'] & A' \arrow[d, "\mu'"] \\ V \arrow[r, "\varphi^V"] & V' \end{tikzcd}$$

Now, suppose that φ is a principal $\mathbb{R}$-bundle embedding. Then, φ^V is also an embedding. In fact, using (2.5), (2.7) and the fact that $\mu \circ \psi_s = \mu$, for all s, we deduce that φ^V is an injective immersion. Moreover, standard topological arguments show that the map $\varphi^V : V \to \varphi^V(V)$ is a homeomorphism.

On the other hand, if $\varphi : A \to A'$ is a diffeomorphism, then φ^V also is a diffeomorphism. Indeed,

$$(\varphi^V)^{-1} = (\varphi^{-1})^{V'}.$$

Definition 2.8. *Let $\mu : (A, \Omega) \to V$ and $\mu' : (A', \Omega') \to V'$ be symplectic principal $\mathbb{R}$-bundles. A smooth function $\varphi : A \to A'$ is said to be a* symplectic principal $\mathbb{R}$-bundle morphism *if φ is a principal $\mathbb{R}$-bundle morphism such that $\varphi^*\Omega' = \Omega$.*

If we change the word "morphism" by "embedding" (respectively, by "isomorphism") in the previous definition, we obtain the notion of a *symplectic principal* $\mathbb{R}$*-bundle embedding* (respectively, a *symplectic principal* $\mathbb{R}$*-bundle isomorphism*).

Since the infinitesimal generators Z_μ and $Z_{\mu'}$ of μ and μ', respectively are φ-related, then one may easily prove the following result which will be useful in the sequel.

Proposition 2.9. *Let $\mu : (A, \Omega) \to V$ and $\mu' : (A', \Omega') \to V'$ be symplectic principal $\mathbb{R}$-bundles with infinitesimal generators Z_μ and $Z_{\mu'}$, respectively. If $\varphi : A \to A'$ is a symplectic principal $\mathbb{R}$-bundle morphism, then*

$$\varphi^*(i_{Z_{\mu'}}\Omega') = i_{Z_\mu}\Omega.$$

In general, if φ is a principal $\mathbb{R}$-bundle morphism, the maps φ and φ^V are not Poisson. In fact, suppose that φ is a symplectic principal $\mathbb{R}$-bundle embedding. In what follows we will identify the tangent space T_aA at a point $a \in A$ (respectively, the tangent space T_vV at $v \in V$) with the subspace $T_a\varphi(T_aA) \subseteq T_{\varphi(a)}A'$ (respectively, $T_v\varphi^V(T_vV) \subseteq T_{\varphi^V(v)}V'$). Under this identification, for all $a \in A$, we have the following decomposition

$$T_{\varphi(a)}A' = T_aA \oplus (T_aA)^{\Omega'}$$

where $(T_aA)^{\Omega'}$ denotes the symplectic orthogonal space of T_aA with respect to the symplectic form $\Omega'_{\varphi(a)}$ on $T_{\varphi(a)}A'$. Note that if $\Lambda_{A'} \in \mathcal{V}^2(A')$ is the Poisson structure associated with the symplectic 2-form Ω', then $(T_aA)^{\Omega'}$ is just the image of the annihilator $(T_aA)^o$ of T_aA in $T_{\varphi(a)}A'$ through $\sharp_{\Lambda_{A'}}$. Therefore,

$$T_{\varphi(a)}A' = T_aA \oplus \sharp_{\Lambda_{A'}}\big((T_aA)^o\big). \tag{2.8}$$

The corresponding splitting of $T^*_{\varphi(a)}A'$ is

$$T^*_{\varphi(a)}A' = (T_aA)^o \oplus \big(\sharp_{\Lambda_{A'}}(T_aA)^o\big)^o. \tag{2.9}$$

The projectors $\widetilde{P}_a : T^*_{\varphi(a)}A' \to (\sharp_{\Lambda_{A'}}(T_aA)^o)^o$ and $\widetilde{Q}_a : T^*_{\varphi(a)}A' \to (T_aA)^o$ correspond with the splitting (2.9).

On the other hand, using (2.7), (2.8) and the fact that the infinitesimal generators Z_μ and $Z_{\mu'}$ are φ-related, one may obtain that

$$T_vV \cap T_{\varphi(a)}\mu'(\sharp_{\Lambda_{A'}}(T_aA)^o) = \{0\}, \qquad \text{with } v = \mu(a) \in V,$$

which implies that

$$T_{\varphi^V(v)}V' = T_vV \oplus T_{\varphi(a)}\mu'(\sharp_{\Lambda_{A'}}(T_aA)^o). \tag{2.10}$$

Moreover, from (2.7) and since $\dim(T_vV)^o = \dim(T_aA)^o$, it follows that

$$(T_aA)^o = T^*_{\varphi(a)}\mu((T_vV)^o),$$

$(T_vV)^o$ being the annihilator of T_vV in $T_{\varphi^V(v)}V'$. Therefore, using (1.4) (see Section 1.1) and the fact that μ' is a Poisson map, we deduce that

$$T_{\varphi(a)}\mu'(\sharp_{\Lambda_{A'}}((T_aA)^o)) = \sharp_{\Lambda_{V'}}(T_vV)^o$$

where $\Lambda_{V'}$ is the Poisson structure on V' induced by the symplectic principal $\mathbb{R}$-bundle $\mu' : (A', \Omega') \to V'$. So, we may again consider the splittings

$$\begin{aligned} T_{\varphi^V(v)}V' &= T_vV \oplus \sharp_{\Lambda_{V'}}(T_vV)^o, \\ T^*_{\varphi^V(v)}V' &= (T_vV)^o \oplus \left(\sharp_{\Lambda_{V'}}(T_vV)^o\right)^o \end{aligned} \tag{2.11}$$

and the projectors

$$\widetilde{p}_v : T^*_{\varphi^V(v)}V' \to \left(\sharp_{\Lambda_{V'}}(T_vV)^o\right)^o, \qquad \widetilde{q}_v : T^*_{\varphi^V(v)}V' \to (T_vV)^o.$$

Now, using these splittings, we may compare the Poisson structure on A and A' (respectively, V and V').

Proposition 2.10. *Let $\varphi : A \to A'$ be an embedding of the principal $\mathbb{R}$-bundles $\mu : (A, \Omega) \to V$ and $\mu' : (A', \Omega') \to V'$ and let $\varphi^V : V \to V'$ be the corresponding embedding between the base spaces V and V'. Then*

1) *The Poisson structures Λ_A and $\Lambda_{A'}$ on A and A' respectively, are related as follows*

$$\begin{aligned} \Lambda_A(a)((T^*_a\varphi)(\alpha'_1), (T^*_a\varphi)(\alpha'_2)) &= \Lambda_{A'}(\varphi(a))(\widetilde{P}_a(\alpha'_1), \widetilde{P}_a(\alpha'_2)) \\ &= \Lambda_{A'}(\varphi(a))(\alpha'_1, \alpha'_2) - \Lambda_{A'}(\varphi(a))(\widetilde{Q}_a(\alpha'_1), \widetilde{Q}_a(\alpha'_2)) \end{aligned} \tag{2.12}$$

with $a \in A$ and $\alpha'_1, \alpha'_2 \in T^*_{\varphi(a)}A'$.

2) *The Poisson structures Λ_V and $\Lambda_{V'}$ induced on V and V' respectively, by the symplectic principal $\mathbb{R}$-bundles μ and μ', are related by*

$$\begin{aligned} \Lambda_V(v)((T^*_v\varphi^V)(\sigma'_1), (T^*_v\varphi^V)(\sigma'_2)) &= \Lambda_{V'}(\varphi^V(v))(\widetilde{p}_v(\sigma'_1), \widetilde{p}_v(\sigma'_2)) \\ &= \Lambda_{V'}(\varphi^V(v))(\sigma'_1, \sigma'_2) - \Lambda_{V'}(\varphi^V(v))(\widetilde{q}_v(\sigma'_1), \widetilde{q}_v(\sigma'_2)) \end{aligned}$$

with $v \in V$ and $\sigma'_1, \sigma'_2 \in T^*_{\varphi^V(v)}V'$.

3) *If $\varphi : A \to A'$ is an isomorphism of principal $\mathbb{R}$-bundles, then $\varphi^V : V \to V'$ is a Poisson isomorphism.*

Proof. 1) If there exists $i \in \{1,2\}$ such that $\alpha'_i \in (T_aA)^o$, then it is clear that

$$(T_a^*\varphi)(\alpha'_i) = 0 \quad \text{and} \quad \widetilde{P}_a(\alpha'_i) = 0$$

and, thus,

$$\Lambda_A(a)((T_a^*\varphi)(\alpha'_1), (T_a^*\varphi)(\alpha'_2)) = \Lambda_{A'}(\varphi(a))(\widetilde{P}_a(\alpha'_1), \widetilde{P}_a(\alpha'_2)) = 0.$$

Therefore, we must prove that if $\alpha'_1, \alpha'_2 \in (\sharp_{\Lambda_{A'}}(T_aA)^o)^o$ then

$$\begin{aligned}\Lambda_A(a)((T_a^*\varphi)(\alpha'_1), (T_a^*\varphi)(\alpha'_2)) &= \Lambda_{A'}(\varphi(a))(\widetilde{P}_a(\alpha'_1), \widetilde{P}_a(\alpha'_2)) \\ &= \Lambda_{A'}(\varphi(a))(\alpha'_1, \alpha'_2).\end{aligned}$$

Now, we have that $(\sharp_{\Lambda_{A'}}(T_aA)^o)^o = (\flat_{\Omega'})_{\varphi(a)}(T_aA)$. This implies that

$$\alpha'_i = i_{X_i}\Omega'_{\varphi(a)}, \qquad \text{with } X_i \in T_aA$$

and

$$\begin{aligned}&\Lambda_A(a)((T_a^*\varphi)(\alpha'_1), (T_a^*\varphi)(\alpha'_2)) \\ &\quad = \Lambda_A(a)((T_a^*\varphi)((\flat_{\Omega'})_{\varphi(a)}(X_1)), (T_a^*\varphi)((\flat_{\Omega'})_{\varphi(a)}(X_2))).\end{aligned}$$

Consequently, using (1.5) and the fact that

$$(\flat_\Omega)_a = T_a^*\varphi \circ (\flat_{\Omega'})_{\varphi(a)}|_{T_aA}$$

it follows that

$$\begin{aligned}\Lambda_A(a)((T_a^*\varphi)(\alpha'_1), (T_a^*\varphi)(\alpha'_2)) &= \Omega_a(X_1, X_2) = \Omega'_{\varphi(a)}(X_1, X_2) \\ &= \Lambda'(\varphi(a))(\alpha'_1, \alpha'_2).\end{aligned}$$

Note that

$$\widetilde{P}_a(\alpha'_i) = \alpha'_i - \widetilde{Q}_a(\alpha'_i), \qquad i \in \{1,2\}$$

and that

$$\Lambda_{A'}(\varphi(a))(\widetilde{P}_a(\beta'), \widetilde{Q}_a(\gamma')) = 0, \qquad \text{for } \beta', \gamma' \in T^*_{\varphi(a)}A'.$$

2) From (2.3), we have that, if $a \in A$ and $\mu(a) = v$, then

$$\begin{aligned}&\Lambda_V(v)((T_v^*\varphi^V)(\sigma'_1), (T_v^*\varphi^V)(\sigma'_2)) \\ &\quad = \Lambda_A(a)((T_a^*(\varphi^V \circ \mu))(\sigma'_1), (T_a^*(\varphi^V \circ \mu))(\sigma'_2))\end{aligned}$$

Thus, using (2.7), we deduce that

$$\begin{aligned}&\Lambda_V(v)((T_v^*\varphi^V)(\sigma'_1), (T_v^*\varphi^V)(\sigma'_2)) \\ &\quad = \Lambda_A(a)((T_a^*\varphi)((T^*_{\varphi(a)}\mu')(\sigma'_1)), (T_a^*\varphi)((T^*_{\varphi(a)}\mu')(\sigma'_2)))\end{aligned}$$

and, from (2.12), it follows that

$$\begin{aligned}&\Lambda_V(v)((T_v^*\varphi^V)(\sigma_1'),(T_v^*\varphi^V)(\sigma_2'))\\&\qquad=\Lambda_{A'}(\varphi(a))(\widetilde{P}_a((T_{\varphi(a)}^*\mu')(\sigma_1')),\widetilde{P}_a((T_{\varphi(a)}^*\mu')(\sigma_2'))).\end{aligned}$$

On the other hand, since $\widetilde{P}_a\circ T_{\varphi(a)}^*\mu'=T_{\varphi(a)}^*\mu'\circ\widetilde{p}_v$, we obtain that

$$\begin{aligned}&\Lambda_V(v)((T_v^*\varphi^V)(\sigma_1'),(T_v^*\varphi^V)(\sigma_2'))\\&\qquad=\Lambda_{A'}(\varphi(a))((T_{\varphi(a)}^*\mu')(\widetilde{p}_v(\sigma_1')),(T_{\varphi(a)}^*\mu')(\widetilde{p}_v(\sigma_2'))).\end{aligned}$$

Finally, using (2.3), we conclude that

$$\Lambda_V(v)((T_v^*\varphi^V)(\sigma_1'),(T_v^*\varphi^V)(\sigma_2'))=\Lambda_{V'}(\varphi^V(v))(\widetilde{p}_v(\sigma_1'),\widetilde{p}_v(\sigma_2')).$$

3) In this case, φ^V is a diffeomorphism and $\widetilde{p}_v=id$. Therefore, using the second part of this proposition, we deduce that φ^V is a Poisson isomorphism. □

2.3 Canonical actions and momentum maps

In this section, we suppose that $\mu:(A,\Omega)\to V$ is a symplectic principal $\mathbb{R}$-bundle with principal action $\psi:\mathbb{R}\times A\to A$.

Let

$$\phi:G\times A\to A,\qquad (g,a)\mapsto\phi_g(a)$$

be an action of a connected Lie group G on the manifold A. We consider a special type of actions which are compatible with μ in a certain sense.

Definition 2.11. *We say that an action $\phi:G\times A\to A$ is a* canonical action *on the symplectic principal $\mathbb{R}$-bundle $\mu:(A,\Omega)\to V$ if the following conditions hold:*

i) *ϕ is a symplectic action,*

ii) *the actions ψ and ϕ commute, that is*

$$\phi_g\circ\psi_s=\psi_s\circ\phi_g,\qquad \textit{for any } g\in G,\ s\in\mathbb{R},\tag{2.13}$$

iii) *the 1-form $\zeta_\mu=i_{Z_\mu}\Omega$ is basic with respect to ϕ, i.e. $\zeta_\mu(\xi_A)=0$ for any $\xi\in\mathfrak{g}$, where ξ_A is the infinitesimal generator of ϕ defined by ξ.*

We will see that if $\phi : G \times A \to A$ is a canonical action on μ with momentum map $J : A \to \mathfrak{g}^*$, then one may induce canonically a Poisson action $\phi^V : G \times V \to V$ on V with a momentum map $J^V : V \to \mathfrak{g}^*$.

In fact, let ξ be an element of the Lie algebra $\mathfrak{g}$. Using the fact that ζ_μ is basic, it follows that

$$Z_\mu(J_\xi) = i_{\xi_A}\Omega(Z_\mu) = -\zeta_\mu(\xi_A) = 0. \tag{2.14}$$

Therefore, the function $J_\xi : A \to \mathbb{R}$ is constant on the fibers of μ. Thus, if $a, b \in A$ are in the same fiber, we have that $J_\xi(a) = J_\xi(b)$. Since ξ is arbitrary, it follows that $J(a) = J(b)$. In other words,

$$J \circ \psi_s = J, \qquad \text{for any } s. \tag{2.15}$$

Using this fact and (2.13), we may define the action $\phi^V : G \times V \to V$ of G on V and the map $J^V : V \to \mathfrak{g}^*$ by

$$\phi^V(g, v) = \mu(\phi_g(a)), \quad J^V(v) = J(a), \qquad \text{for any } g \in G,\ v \in V \tag{2.16}$$

where $a \in A$ is an arbitrary element in $\mu^{-1}(v)$.

By construction, μ is equivariant with respect to the actions ϕ and ϕ^V, i.e.

$$\phi^V_g \circ \mu = \mu \circ \phi_g, \qquad \text{for any } g \in G \tag{2.17}$$

and

$$J^V \circ \mu = J. \tag{2.18}$$

So, μ transforms the infinitesimal generator $\xi_A \in \mathfrak{X}(A)$ of $\xi \in \mathfrak{g}$ with respect to the action ϕ into the infinitesimal generator $\xi_V \in \mathfrak{X}(V)$ of ξ with respect to the action ϕ^V, that is,

$$T_a\mu(\xi_A(a)) = \xi_V(\mu(a)), \qquad \text{for any } a \in A. \tag{2.19}$$

Proposition 2.12. *If $\phi : G \times A \to A$ is a canonical action equipped with a momentum map $J : A \to \mathfrak{g}^*$, then:*

i) *$\phi^V : G \times V \to V$ is a Poisson action;*

ii) *$J^V : V \to \mathfrak{g}^*$ is a momentum map associated with ϕ^V and, if J is Ad^*-equivariant, then so is J^V.*

Proof. Let $g \in G$. The diffeomorphism $\phi_g : A \to A$ is symplectic and so is a Poisson map. Using this fact, (2.3) and (2.17), one has that for any $f, f' \in C^\infty(V)$

$$\left\{f \circ \phi^V_g, f' \circ \phi^V_g\right\}_V \circ \mu = \{f, f'\}_V \circ \phi^V_g \circ \mu.$$

Since μ is surjective, it follows that $\phi_g^V : V \to V$ is a Poisson map.

Now, we prove that J^V is a momentum map, that is, $\xi_V = \mathcal{H}_{J_\xi^V}$, for any $\xi \in \mathfrak{g}$. In fact, for any $f \in C^\infty(V)$ and $a \in A$, one has, from (2.3) and (2.19), that

$$\begin{aligned}
(\xi_V(f))\,(\mu(a)) &= (T_a\mu(\xi_A(a)))\,(f) = \xi_A(a)(f \circ \mu) \\
&= \mathcal{H}_{J_\xi}(a)(f \circ \mu) = \{f \circ \mu, J_\xi\}_A\,(a) \\
&= \left\{f \circ \mu, J_\xi^V \circ \mu\right\}_A(a) = \left\{f, J_\xi^V\right\}_V(\mu(a)) \\
&= (\mathcal{H}_{J_\xi^V}(f))(\mu(a)).
\end{aligned}$$

Since a is an arbitrary element of A and μ is surjective, we obtain that $\xi_V = \mathcal{H}_{J_\xi^V}$.

If J is Ad^*-equivariant, then for any $v = \mu(a) \in V$ and, for any $g \in G$,

$$\mathrm{Ad}^*_{g^{-1}}(J^V(v)) = \mathrm{Ad}^*_{g^{-1}}(J(a)) = J(\phi_g(a)) = J^V(\phi_g^V(v)).$$

Thus, J^V is Ad^*-equivariant. □

2.4 Reduction Theorem of symplectic principal $\mathbb{R}$-bundles

In this section, we will use the results of Section 1.5 to reduce a symplectic principal $\mathbb{R}$-bundle equipped with a canonical action and an Ad^*-equivariant momentum map.

Suppose that $\mu : (A, \Omega) \to V$ is a symplectic principal $\mathbb{R}$-bundle equipped with a canonical action $\phi : G \times A \to A$ of a Lie group G with an Ad^*-equivariant momentum map $J : A \to \mathfrak{g}^*$. One may induce a Poisson action $\phi^V : G \times V \to V$ on V with an Ad^*-equivariant momentum map $J^V : V \to \mathfrak{g}^*$. Assume that ϕ^V is free and proper. Then, so is ϕ.

Since ϕ is free, J is a submersion and, from (2.18), J^V is also a submersion. Consequently, any element of $\mathfrak{g}^*$ is a regular value for J and J^V.

Let $\nu \in \mathfrak{g}^*$. From Marsden-Weinstein reduction Theorem (respectively, Poisson reduction Theorem), we may induce a reduced symplectic structure Ω_ν (respectively, a reduced Poisson bracket $\{\cdot,\cdot\}_\nu$) on the quotient space $A_\nu = J^{-1}(\nu)/G_\nu$ (respectively, $V_\nu = (J^V)^{-1}(\nu)/G_\nu$). Let's prove that A_ν and V_ν are the total space and the base manifold, respectively, of a reduced principal $\mathbb{R}$-bundle $\mu_\nu : A_\nu \to V_\nu$.

The map $\mu_\nu : A_\nu \to V_\nu$ is defined as follows. Using (2.18), it follows that the restriction $\mu : J^{-1}(\nu) \to (J^V)^{-1}(\nu)$ of μ to the closed submanifold $J^{-1}(\nu)$ is a surjective submersion. Moreover, we have that the actions $\phi :$

$G_\nu \times J^{-1}(\nu) \to J^{-1}(\nu)$ and $\phi^V : G_\nu \times (J^V)^{-1}(\nu) \to (J^V)^{-1}(\nu)$ of the isotropy group G_ν on $J^{-1}(\nu)$ and $(J^V)^{-1}(\nu)$ respectively, are free and proper and μ is equivariant with respect to them. Denote by

$$\mu_\nu : A_\nu = J^{-1}(\nu)/G_\nu \to V_\nu = (J^V)^{-1}(\nu)/G_\nu$$

the induced map on the quotient spaces which is characterized by

$$\mu_\nu \circ \pi_\nu = \pi_\nu^V \circ \mu, \tag{2.20}$$

where $\pi_\nu : J^{-1}(\nu) \to A_\nu$ and $\pi_\nu^V : (J^V)^{-1}(\nu) \to V_\nu$ are the corresponding canonical projections. Note that μ_ν is a surjective submersion.

Moreover, using (2.15), we have that the principal action $\psi : \mathbb{R} \times A \to A$ restricts to an action of $\mathbb{R}$ on $J^{-1}(\nu)$. Then, we may define

$$\psi_\nu : \mathbb{R} \times A_\nu \to A_\nu, \qquad \psi_\nu(s, \pi_\nu(a)) = \pi_\nu(\psi(s, a)). \tag{2.21}$$

Using (2.13), it's easy to show that this map is well-defined and that it is an action of the Lie group $\mathbb{R}$ on A_ν.

Note that the map $\pi_\nu : J^{-1}(\nu) \to A_\nu$ is equivariant with respect to the restricted action $\mathbb{R} \times J^{-1}(\nu) \to J^{-1}(\nu)$ and to the action $\psi_\nu : \mathbb{R} \times A_\nu \to A_\nu$.

Now, we may prove the reduction theorem for symplectic principal $\mathbb{R}$-bundles.

Theorem 2.13. *Let $\mu : (A, \Omega) \to V$ be a symplectic principal $\mathbb{R}$-bundle equipped with a canonical action $\phi : G \times A \to A$ and an Ad^*-equivariant momentum map $J : A \to \mathfrak{g}^*$. Suppose that the induced action $\phi^V : G \times V \to V$ is free and proper. Then, for any $\nu \in \mathfrak{g}^*$, $\mu_\nu : (A_\nu, \Omega_\nu) \to V_\nu$ is a symplectic principal $\mathbb{R}$-bundle with principal action defined by (2.21), where Ω_ν is the reduced symplectic structure on the reduced space $A_\nu = J^{-1}(\nu)/G_\nu$. Moreover, the restriction of the infinitesimal generator Z_μ of μ to $J^{-1}(\nu)$ is tangent to $J^{-1}(\nu)$ and π_ν-projectable. Its π_ν-projection is the infinitesimal generator Z_{μ_ν} of μ_ν.*

Proof. First of all, we will see that ψ_ν is a free action. Indeed, suppose that $(\psi_\nu)_s(\pi_\nu(a)) = \pi_\nu(a)$, for $a \in J^{-1}(\nu)$. Then, from (2.13) and (2.21), we deduce that there exists $g \in G_\nu$ such that

$$a = \psi_s(\phi_g(a)). \tag{2.22}$$

This implies that

$$\mu(a) = \mu(\psi_s(\phi_g(a))) = \mu(\phi_g(a))$$

and, using (2.17), it follows that

$$\phi^V_g(\mu(a)) = \mu(a).$$

Thus, since ϕ^V is a free action, we obtain that $g = e$. Therefore, from (2.22), we conclude that $s = 0$.

Next, we will prove that the fibers of μ_ν are just the orbits of the action of $\mathbb{R}$ on $A_\nu = J^{-1}(\nu)/G_\nu$. In other words, we will see that

$$(\psi_\nu)_{\pi_\nu(a)}(\mathbb{R}) = (\mu_\nu)^{-1}(\mu_\nu(\pi_\nu(a))), \qquad \text{for } a \in J^{-1}(\nu).$$

In fact, using (2.20) and (2.21), we deduce that

$$\mu_\nu((\psi_\nu)_s(\pi_\nu(a))) = \mu_\nu(\pi_\nu(a)), \qquad \text{for } s \in \mathbb{R}.$$

So, we have that

$$(\psi_\nu)_{\pi_\nu(a)}(\mathbb{R}) \subseteq (\mu_\nu)^{-1}(\mu_\nu(\pi_\nu(a))), \qquad \text{for } a \in J^{-1}(\nu).$$

On the other hand, if $b \in J^{-1}(\nu)$ and

$$\mu_\nu(\pi_\nu(b)) = \mu_\nu(\pi_\nu(a))$$

then, from (2.17) and (2.20), it follows that there exist $s \in \mathbb{R}$ and $g \in G_\nu$ satisfying

$$b = \psi_s(\phi_g(a)).$$

Thus, using (2.21), we conclude that $(\psi_\nu)_s(\pi_\nu(a)) = \pi_\nu(b)$. This proves that

$$(\mu_\nu)^{-1}(\mu_\nu(\pi_\nu(a))) \subseteq (\psi_\nu)_{\pi_\nu(a)}(\mathbb{R}).$$

Consequently, $\mu_\nu : A_\nu \to V_\nu$ is a principal $\mathbb{R}$-bundle.

Now, as we know, the action $\psi : \mathbb{R} \times A \to A$ restricts to an action of $\mathbb{R}$ on $J^{-1}(\nu)$. This implies that the restriction to $J^{-1}(\nu)$ of the infinitesimal generator Z_μ of μ is tangent to $J^{-1}(\nu)$ and $Z_{\mu|J^{-1}(\nu)}$ is just the infinitesimal generator of the action of $\mathbb{R}$ on $J^{-1}(\nu)$.

In addition, since the projection π_ν is equivariant, we obtain that $Z_{\mu|J^{-1}(\nu)}$ is π_ν-projectable and its projection is the infinitesimal generator Z_{μ_ν} of μ_ν.

Finally, we prove that Z_{μ_ν} is a locally Hamiltonian vector field. We will show that the flow $(\psi_\nu)_s : A_\nu \to A_\nu$ of Z_{μ_ν} preserves the symplectic form Ω_ν. In fact, from the equivariance of π_ν, it follows that

$$(\psi_\nu)_s \circ \pi_\nu = \pi_\nu \circ \psi_s. \tag{2.23}$$

Thus,

$$\pi_\nu^* \left((\psi_\nu)_s^* \Omega_\nu\right) = \psi_s^*(\pi_\nu^* \Omega_\nu) = \psi_s^*(i_\nu^* \Omega) = i_\nu^* \Omega = \pi_\nu^* \Omega_\nu,$$

where we have used (1.21) and the invariance of Ω under the action of ψ_s. As a consequence, we have that $(\psi_\nu)_s^* \Omega_\nu = \Omega_\nu$. □

From Proposition 2.6, the symplectic 2-form Ω_ν on A_ν induces a Poisson structure $\{\cdot,\cdot\}_{V_\nu}$ on V_ν. On the other hand, using Theorem 1.8, we have that V_ν is equipped with a reduced Poisson structure. The following result proves that these structures are the same one.

Proposition 2.14. *Under the same hypotheses as in Theorem 2.13, the reduced Poisson bracket $\{\cdot,\cdot\}_\nu$ on V_ν is just the one induced by the symplectic principal $\mathbb{R}$-bundle $\mu_\nu : A_\nu \to V_\nu$.*

Proof. Let f_ν, f'_ν be functions on V_ν and $\pi^V_\nu(v) \in V_\nu$, with $v \in (J^V)^{-1}(\nu)$. Choose $a \in J^{-1}(\nu)$ such that $\mu(a) = v$. The bracket $\{\cdot,\cdot\}_\nu$ is characterized by

$$\{f_\nu, f'_\nu\}_\nu (\pi^V_\nu(v)) = \{f, f'\}_V (v),$$

where $f, f' \in C^\infty(V)$ are arbitrary G-invariant extensions of $f_\nu \circ \pi^V_\nu$ and $f'_\nu \circ \pi^V_\nu$, respectively.

Note that $f \circ \mu, f' \circ \mu \in C^\infty(A)$ are G-invariant extensions of $f_\nu \circ \pi^V_\nu \circ \mu_{|J^{-1}(\nu)}$ and $f'_\nu \circ \pi^V_\nu \circ \mu_{|J^{-1}(\nu)}$, respectively. Applying Theorem 1.9, we obtain that the Poisson bracket $\{\cdot,\cdot\}_{A_\nu}$ on A_ν induced by Ω_ν may be expressed as follows

$$\{f_\nu \circ \mu_\nu, f'_\nu \circ \mu_\nu\}_{A_\nu} (\pi_\nu(a)) = \{f \circ \mu, f' \circ \mu\}_A (a).$$

Therefore, using (2.3) refered to μ and μ_ν, we have

$$\begin{aligned} \{f_\nu, f'_\nu\}_{V_\nu} (\pi^V_\nu(v)) &= \{f_\nu \circ \mu_\nu, f'_\nu \circ \mu_\nu\}_{A_\nu} (\pi_\nu(a)) \\ &= \{f \circ \mu, f' \circ \mu\}_A (a) \\ &= \{f_\nu, f'_\nu\}_\nu (\pi^V_\nu(v)). \end{aligned} \tag{2.24}$$

This proves that $\{f_\nu, f'_\nu\}_{V_\nu} = \{f_\nu, f'_\nu\}_\nu$. $\square$

2.5 Orbit reduction Theorem

In this section, we will give an other description of the reduced symplectic principal $\mathbb{R}$-bundle in term of the orbit of the coadjoint action.

Let $\mu : (A, \Omega) \to V$ be a symplectic principal $\mathbb{R}$-bundle with principal $\mathbb{R}$-action $\psi : \mathbb{R} \times A \to A$. If $\phi : G \times A \to A$ is a canonical action of a connected Lie group G on μ, $J : A \to \mathfrak{g}^*$ an Ad^*-equivariant momentum map and $\nu \in \mathfrak{g}^*$, then, under regularity conditions, we may consider the reduced symplectic principal $\mathbb{R}$-bundle $\mu_\nu : (A_\nu, \Omega_\nu) \to V_\nu$ (see Theorem 2.13).

On the other hand, the total space $A_\nu = J^{-1}(\nu)/G_\nu$ (respectively, the base space V_ν) of μ_ν is diffeomorphic to the quotient space $A_{\mathcal{O}} = J^{-1}(\mathcal{O})/G$

(respectively, $V_{\mathcal{O}} = (J^V)^{-1}(\mathcal{O})/G$). Here $J^V : V \to \mathfrak{g}^*$ denotes the Ad^*-equivariant momentum map with respect to the Poisson action ϕ^V of G on V. The diffeomorphism between A_ν and $A_{\mathcal{O}}$ is explicitely given by

$$[\iota] : J^{-1}(\nu)/G_\nu \to J^{-1}(\mathcal{O})/G, \qquad \pi_\nu(a) \mapsto \pi_{\mathcal{O}}(a),$$

where $\pi_\nu : J^{-1}(\nu) \to A_\nu$ and $\pi_{\mathcal{O}} : J^{-1}(\mathcal{O}) \to A_{\mathcal{O}}$ are the corresponding projections.

Notice that $A_{\mathcal{O}}$ is equipped with a symplectic 2-form $\Omega_{\mathcal{O}}$ characterized by

$$\pi_{\mathcal{O}}^*\Omega_{\mathcal{O}} = i_{\mathcal{O}}^*\Omega + J_{\mathcal{O}}^*\widehat{\Omega}, \tag{2.25}$$

where $J_{\mathcal{O}} : J^{-1}(\mathcal{O}) \to \mathcal{O}$ is the restriction of J to $J^{-1}(\mathcal{O})$, $i_{\mathcal{O}} : J^{-1}(\mathcal{O}) \to A$ is the inclusion, and $\widehat{\Omega}$ is the Kirillov-Kostant-Souriau symplectic 2-form on $\mathcal{O}$. In fact, $[\iota] : (A_\nu, \Omega_\nu) \to (A_{\mathcal{O}}, \Omega_{\mathcal{O}})$ is a symplectomorphism (see Section 1.5.2).

In what follows, we will show that $A_{\mathcal{O}}$ is the total space of a symplectic principal $\mathbb{R}$-bundle $\mu_{\mathcal{O}} : (A_{\mathcal{O}}, \Omega_{\mathcal{O}}) \to V_{\mathcal{O}}$.

The map $\mu_{\mathcal{O}} : A_{\mathcal{O}} \to V_{\mathcal{O}}$ is defined as follows. The restriction $\mu : J^{-1}(\mathcal{O}) \to (J^V)^{-1}(\mathcal{O})$ of μ to the submanifold $J^{-1}(\mathcal{O})$ is a G-equivariant surjective submersion. By passing to the quotient, we obtain a surjective submersion

$$\mu_{\mathcal{O}} : A_{\mathcal{O}} = J^{-1}(\mathcal{O})/G \to V_{\mathcal{O}} = (J^V)^{-1}(\mathcal{O})/G$$

characterized by

$$\mu_{\mathcal{O}} \circ \pi_{\mathcal{O}} = \pi_{\mathcal{O}}^V \circ \mu, \tag{2.26}$$

where $\pi_{\mathcal{O}} : J^{-1}(\mathcal{O}) \to A_{\mathcal{O}}$ and $\pi_{\mathcal{O}}^V : (J^V)^{-1}(\mathcal{O}) \to V_{\mathcal{O}}$ are the corresponding projections. Moreover, we may define a principal $\mathbb{R}$-action on $A_{\mathcal{O}}$ using the restricted action $\mathbb{R} \times J^{-1}(\mathcal{O}) \to J^{-1}(\mathcal{O})$ and by passing to the quotient. Precisely, we define

$$\psi_{\mathcal{O}} : \mathbb{R} \times A_{\mathcal{O}} \to A_{\mathcal{O}}, \qquad \psi_{\mathcal{O}}(s, \pi_{\mathcal{O}}(a)) = \pi_{\mathcal{O}}(\psi(s,a)). \tag{2.27}$$

As in the previous section for $\psi_\nu : \mathbb{R} \times A_\nu \to A_\nu$, one may prove that $\psi_{\mathcal{O}}$ is well-defined and that it is an action of the Lie group $\mathbb{R}$ on $A_{\mathcal{O}}$.

Note that the map $\pi_{\mathcal{O}} : J^{-1}(\mathcal{O}) \to A_{\mathcal{O}}$ is equivariant with respect to the restricted action $\mathbb{R} \times J^{-1}(\mathcal{O}) \to J^{-1}(\mathcal{O})$ and to the action $\psi_{\mathcal{O}} : \mathbb{R} \times A_{\mathcal{O}} \to A_{\mathcal{O}}$.

Now, we may prove the orbit version of the reduction theorem for symplectic principal $\mathbb{R}$-bundles.

Theorem 2.15. *Let $\mu : (A, \Omega) \to V$ be a symplectic principal $\mathbb{R}$-bundle equipped with a canonical action $\phi : G \times A \to A$ and an Ad^*-equivariant* *momentum map $J : A \to \mathfrak{g}^*$. Suppose that the induced action $\phi^V : G\times V \to V$ is free and proper. Then, for any $\nu \in \mathfrak{g}^*$, $\mu_{\mathcal{O}} : (A_{\mathcal{O}}, \Omega_{\mathcal{O}}) \to V_{\mathcal{O}}$ is a symplectic principal $\mathbb{R}$-bundle with principal action defined by (2.27).*

Moreover, the restriction of the infinitesimal generator Z_μ of μ to $J^{-1}(\mathcal{O})$ is tangent to $J^{-1}(\mathcal{O})$ and $\pi_{\mathcal{O}}$-projectable. Its $\pi_{\mathcal{O}}$-projection is the infinitesimal generator $Z_{\mu_{\mathcal{O}}}$ of $\mu_{\mathcal{O}}$.

Proof. Proceeding as in the proof of Theorem 2.13, we deduce that $\psi_{\mathcal{O}}$ is a free action and that the orbits of $\psi_{\mathcal{O}}$ are just to the fibers of $\mu_{\mathcal{O}}$. Now, we will show that $\psi_{\mathcal{O}}$ is a symplectic action, i.e. $(\psi_{\mathcal{O}})^*_s\Omega_{\mathcal{O}} = \Omega_{\mathcal{O}}$ for any $s \in \mathbb{R}$. Equivalently, using (2.25), it is sufficient to prove that

$$\pi^*_{\mathcal{O}}((\psi_{\mathcal{O}})^*_s\Omega_{\mathcal{O}}) = i^*_{\mathcal{O}}\Omega + J^*_{\mathcal{O}}\widehat{\Omega}.$$

Let's compute the left hand side. Using the G-equivariance of $\pi_{\mathcal{O}}$ and (2.25), we get

$$\pi^*_{\mathcal{O}}((\psi_{\mathcal{O}})^*_s\Omega_{\mathcal{O}}) = \psi^*_s(\pi^*_{\mathcal{O}}\Omega_{\mathcal{O}}) = \psi^*_s(i^*_{\mathcal{O}}\Omega + J^*_{\mathcal{O}}\widehat{\Omega}).$$

Now, we use the relations $J_{\mathcal{O}} \circ \psi_s = J_{\mathcal{O}}$ and $i_{\mathcal{O}} \circ \psi_s = \psi_s \circ i_{\mathcal{O}}$. Since ψ_s preserves the symplectic 2-form Ω, we obtain that

$$\pi^*_{\mathcal{O}}((\psi_{\mathcal{O}})^*_s\Omega_{\mathcal{O}}) = i^*_{\mathcal{O}}(\psi^*_s\Omega) + J^*_{\mathcal{O}}\widehat{\Omega} = i^*_{\mathcal{O}}\Omega + J^*_{\mathcal{O}}\widehat{\Omega}.$$

□

The symplectic principal $\mathbb{R}$-bundle $\mu_{\mathcal{O}} : (A_{\mathcal{O}}, \Omega_{\mathcal{O}}) \to V_{\mathcal{O}}$ is another representation of the reduced symplectic principal $\mathbb{R}$-bundle $\mu_\nu : (A_\nu, \Omega_\nu) \to V_\nu$. In fact, as the following result shows, they are isomorphic.

Theorem 2.16. *Let $\mu : (A, \Omega) \to V$ be a symplectic principal $\mathbb{R}$-bundle equipped with a canonical action $\phi : G \times A \to A$ and an Ad^*-equivariant momentum map $J : A \to \mathfrak{g}^*$. Suppose that the induced action $\phi^V : G \times V \to V$ is free and proper. If $\nu \in \mathfrak{g}^*$, denote by $\mu_\nu : (A_\nu, \Omega_\nu) \to V_\nu$ and $\mu_{\mathcal{O}} : (A_{\mathcal{O}}, \Omega_{\mathcal{O}}) \to V_{\mathcal{O}}$ the reduced symplectic principal $\mathbb{R}$-bundles given by Theorems 2.13 and 2.15, respectively. Then, the map*

$$[\iota] : A_\nu \to A_{\mathcal{O}}, \qquad \pi_\nu(a) \mapsto \pi_{\mathcal{O}}(a)$$

is a symplectic principal $\mathbb{R}$-bundle isomorphism.

Proof. As we know (see Theorem 1.14), the map ι is a symplectomorphism. Hence, it is sufficient to show that ι is equivariant with respect to the principal $\mathbb{R}$-actions ψ_ν and $\psi_{\mathcal{O}}$ on A_ν and $A_{\mathcal{O}}$, respectively.

First of all, note that if we denote by $\iota : J^{-1}(\nu) \to J^{-1}(\mathcal{O})$ the inclusion, then the following relation holds

$$[\iota] \circ \pi_\nu = \pi_{\mathcal{O}} \circ \iota.$$

Thus, using the equivariance of π_ν, $\pi_{\mathcal{O}}$ and ι with respect to the corresponding actions, we have

$$\begin{aligned}[\iota] \circ (\psi_\nu)_s \circ \pi_\nu &= [\iota] \circ \pi_\nu \circ \psi_s = \pi_{\mathcal{O}} \circ \iota \circ \psi_s \\ &= \pi_{\mathcal{O}} \circ \psi_s \circ \iota = (\psi_{\mathcal{O}})_s \circ \pi_{\mathcal{O}} \circ \iota \\ &= (\psi_{\mathcal{O}})_s \circ [\iota] \circ \pi_\nu.\end{aligned}$$

Since π_ν is surjective, we obtain that $[\iota]$ is equivariant. □

The base map $[\iota]^V : V_\nu \to V_{\mathcal{O}}$ of the symplectic principal bundle isomorphism $[\iota] : A_\nu \to A_{\mathcal{O}}$ is a Poisson isomorphism between the Poisson manifold V_ν and $V_{\mathcal{O}}$ (see Proposition 2.10). Note that $[\iota]^V : V_\nu \to V_{\mathcal{O}}$ is characterized by the following condition

$$[\iota]^V \circ \pi_\nu^V = \pi_{\mathcal{O}}^V \circ \iota^V, \tag{2.28}$$

where $\pi_\nu^V : (J^V)^{-1}(\nu) \to V_\nu$ and $\pi_{\mathcal{O}}^V : (J^V)^{-1}(\mathcal{O}) \to V_{\mathcal{O}}$ are the canonical projections.

On the other hand, using Proposition 2.14, we have that the Poisson bracket on V_ν is just the reduced Poisson bracket by the Poisson action of G on V. Thus, from Proposition 2.6, we conclude that

Corollary 2.17. *Under the hypotheses of Theorem 2.15, let $\{\cdot,\cdot\}_{V_{\mathcal{O}}}$ the Poisson structure on $V_{\mathcal{O}}$ induced on the base space of the symplectic principal $\mathbb{R}$-bundle $\mu_{\mathcal{O}} : (A_{\mathcal{O}}, \Omega_{\mathcal{O}}) \to V_{\mathcal{O}}$. Then, for any $f_{\mathcal{O}}, h_{\mathcal{O}} \in C^\infty(V_{\mathcal{O}})$, we have that*

$$\{f_{\mathcal{O}}, h_{\mathcal{O}}\}_{V_{\mathcal{O}}} (\pi_{\mathcal{O}}^V(v)) = \{f, h\}_V (v), \qquad \text{for } v \in (J^V)^{-1}(\mathcal{O}),$$

where $f, h \in C^\infty(V)$ are arbitrary G-invariant extensions of $f_{\mathcal{O}} \circ \pi_{\mathcal{O}}^V, h_{\mathcal{O}} \circ \pi_{\mathcal{O}}^V$, respectively. Here $\{\cdot,\cdot\}_V$ denotes the Poisson structure on V induced by the symplectic principal $\mathbb{R}$-bundle $\mu : (A, \Omega) \to V$.

Chapter 3

Standard symplectic principal $\mathbb{R}$-bundles, cotangent lift actions and reduction

3.1 Standard symplectic principal $\mathbb{R}$-bundles and cotangent lift actions

Let $\pi : M \to \mathbb{R}$ be a surjective submersion, Ω_M the canonical symplectic structure on T^*M and $\mu_\pi : (T^*M, \Omega_M) \to V^*\pi$ the standard symplectic principal $\mathbb{R}$-bundle associated with π (see Section 2.1 and Remark 2.4).

Suppose that $\phi : G \times M \to M$ is an action of a connected Lie group G on the manifold M. The lifted action $T^*\phi : G \times T^*M \to T^*M$ is symplectic with respect to the standard symplectic structure Ω_M on T^*M and it admits the Ad^*-equivariant momentum map $J^{T^*M} : T^*M \to \mathfrak{g}^*$ given by (1.50).

The following result gives a sufficient condition for $T^*\phi$ to be a canonical action on the standard symplectic principal $\mathbb{R}$-bundle μ_π.

Proposition 3.1. *Let $\pi : M \to \mathbb{R}$ be a surjective submersion. Denote by $\mu_\pi : (T^*M, \Omega_M) \to V^*\pi$ the corresponding standard symplectic principal $\mathbb{R}$-bundle and by $T^*\phi : G \times T^*M \to T^*M$ the cotangent lift of an action $\phi : G \times M \to M$ of a connected Lie group G on M. If π is G-invariant, i.e. $\pi \circ \phi_g = \pi$ for any $g \in G$, then $T^*\phi$ is a canonical action on μ_π.*

Proof. Recall that the infinitesimal generator ξ_{T^*M} of the action $T^*\phi$ associated to an element ξ of $\mathfrak{g}$ is just the Hamiltonian vector field of the linear function $\widehat{\xi_M} \in C^\infty(T^*M)$ associated with $\xi_M \in \mathfrak{X}(M)$.

Moreover, since $\pi_M : T^*M \to M$ is equivariant with respect to the actions $T^*\phi$ and ϕ, the vector fields ξ_{T^*M} and ξ_M are π_M-related. Now, using the fact that Z_{μ_π} is the Hamiltonian vector field of the function $-\pi \circ \pi_M$, we get

$$\begin{aligned}\Omega_M(\xi_{T^*M}, Z_{\mu_\pi}) &= \left\{\pi \circ \pi_M, \widehat{\xi}_M\right\}_{T^*M} = \mathcal{H}_{\widehat{\xi}_M}(\pi \circ \pi_M) \\ &= \xi_{T^*M}(\pi \circ \pi_M) = \xi_M(\pi) \circ \pi_M = 0,\end{aligned}$$

where the last equality follows from the G-invariance of π. Thus, the 1-form $\zeta_{\mu_\pi} = i_{Z_{\mu_\pi}}\Omega_M$ is basic.

It follows also that

$$[Z_{\mu_\pi}, \xi_{T^*M}] = -[\mathcal{H}_{\pi\circ\pi_M}, \mathcal{H}_{\widehat{\xi}_M}] = \mathcal{H}_{\left\{\pi\circ\pi_M, \widehat{\xi}_M\right\}_{T^*M}} = 0$$

for all $\xi \in \mathfrak{g}$. Since G is connected, the actions ψ and $T^*\phi$ commute. □

Denote by $(T^*\phi)^{V^*\pi} : G \times V^*\pi \to V^*\pi$ and $J^{V^*\pi} : V^*\pi \to \mathfrak{g}^*$ the corresponding action and momentum map, respectively, on $V^*\pi$. Note that $(T^*\phi)^{V^*\pi}$ and $J^{V^*\pi}$ are explicitly given by

$$(T^*\phi)_g^{V^*\pi}(\bar{\alpha}_x)(X) = \bar{\alpha}_x(T_{\phi_g(x)}\phi_{g^{-1}}(X)), \tag{3.1}$$

$$J^{V^*\pi}(\bar{\alpha}_x)(\xi) = \bar{\alpha}_x(\xi_M(x)), \tag{3.2}$$

for any $g \in G$, $\bar{\alpha}_x \in V_x^*\pi$, $X \in V_{\phi_g(x)}\pi$ and $\xi \in \mathfrak{g}$. Moreover, if ϕ is free and proper, so is $(T^*\phi)^{V^*\pi}$.

3.2 Poisson manifolds of corank 1 whose symplectic leaves are isomorphic to a coadjoint orbit

In Chapter 1, we have seen that from Marsden-Weinstein symplectic reduction theorem one may easily deduce that the coadjoint orbits in the dual space of the Lie algebra of a Lie group admit a symplectic structure.

Next, using some results in Chapter 2 on reduction of symplectic principal $\mathbb{R}$-bundles, we will show a version of the Kirillov-Kostant-Souriau theorem for the space of orbits of a free and proper action of G_ν on the total space of a principal G-bundle over the real line.

Suppose that $\phi : G \times M \to M$ is a free and proper action of a connected Lie group G on M such that the space of orbits M/G is diffeomorphic to $\mathbb{R}$. Denote by $\pi : M \to M/G \simeq \mathbb{R}$ the corresponding principal G-bundle projection and let $\mu_\pi : T^*M \to V^*\pi$ be the standard symplectic principal $\mathbb{R}$-bundle associated with π. If $\nu \in \mathfrak{g}^*$ then, applying Theorem 2.13 and using the canonical action $T^*\phi : G \times T^*M \to T^*M$ and the Ad^*-equivariant momentum map $J^{T^*M} : T^*M \to \mathfrak{g}^*$ given by (1.50), we obtain a reduced symplectic principal $\mathbb{R}$-bundle

$$(\mu_\pi)_\nu : ((T^*M)_\nu, (\Omega_M)_\nu) \to (V^*\pi)_\nu$$

with total space the reduced symplectic manifold $((T^*M)_\nu, (\Omega_M)_\nu)$ and base space the reduced Poisson manifold $(V^*\pi)_\nu$ of corank 1.

On the other hand, we have that the vertical bundle $V\pi$ to π is isomorphic to the trivial vector bundle $M \times \mathfrak{g} \to M$, where $\mathfrak{g}$ is the Lie algebra of G. An isomorphism is given by

$$M \times \mathfrak{g} \to V\pi, \qquad (x, \xi) \mapsto \xi_M(x).$$

Under the identification $V^*\pi \simeq M \times \mathfrak{g}^*$, the Poisson bracket $\{\cdot,\cdot\}_{V^*\pi}$ on $V^*\pi$ is characterized by the following relations

$$\begin{aligned} &\left\{\widehat{\xi}, \widehat{\xi'}\right\}_{V^*\pi} = -\widehat{[\xi, \xi']}, \\ &\left\{f \circ pr_1, \widehat{\xi'}\right\}_{V^*\pi} = \xi'_M(f) \circ pr_1, \\ &\{f \circ pr_1, f' \circ pr_1\}_{V^*\pi} = 0, \end{aligned} \tag{3.3}$$

for $\xi, \xi' \in \mathfrak{g}$ and $f, f' \in C^\infty(M)$. Here, if $\eta \in \mathfrak{g}$ then $\widehat{\eta}$ is the linear function on $M \times \mathfrak{g}^*$ defined by

$$\widehat{\eta}(x, \alpha) = \alpha(\eta), \qquad \text{for } (x, \alpha) \in M \times \mathfrak{g}^*.$$

The action $(T^*\phi)^{V^*\pi}$ of G on $V^*\pi \simeq M \times \mathfrak{g}^*$ is given by

$$(T^*\phi)^{V^*\pi}(g, (x, \alpha)) = (\phi_g(x), \mathrm{Ad}^*_{g^{-1}}\alpha), \tag{3.4}$$

for $g \in G$ and $(x, \alpha) \in M \times \mathfrak{g}^*$, and the Ad^*-equivariant momentum map $J^{V^*\pi} : V^*\pi \simeq M \times \mathfrak{g}^* \to \mathfrak{g}^*$ is just

$$J^{V^*\pi}(x, \alpha) = \alpha. \tag{3.5}$$

Therefore,

$$(J^{V^*\pi})^{-1}(\nu) \simeq M \times \{\nu\}$$

which implies that the reduced Poisson manifold $(J^{V^*\pi})^{-1}(\nu)/G_\nu$ is diffeomorphic to the space of orbits M/G_ν of the action of G_ν on M.

Note that, since the action of G_ν is free and proper, then M/G_ν is a smooth manifold. Now we describe this Poisson structure on M/G_ν.

Theorem 3.2. *Let $\phi : G\times M \to M$ be a free and proper action of a connected Lie group G on M such that the space of orbits M/G is diffeomorphic to the real line $\mathbb{R}$. Suppose that $\mathfrak{g}$ is the Lie algebra of G and that $\nu \in \mathfrak{g}^*$. Then, the space of orbits M/G_ν of the action of G_ν on M admits a Poisson structure $\{\cdot,\cdot\}_{M/G_\nu}$ which is isomorphic to the Poisson structure of the base space of the reduced symplectic principal $\mathbb{R}$-bundle $(\mu_\pi)_\nu : ((T^*M)_\nu, (\Omega_M)_\nu) \to (V^*\pi)_\nu$. Moreover,*

i) If $\pi^*_{1,\nu} : M/G_\nu \to \mathbb{R}$ is the canonical projection and $F_{[x]} = (\pi^*_{1,\nu})^{-1}(\pi^*_{1,\nu}[x])$ is the fiber of $\pi^*_{1,\nu}$ at a point $[x] \in M/G_\nu$, with $x \in M$, then the symplectic leaf of M/G_ν passing through $[x]$ is the fiber $F_{[x]}$. The map

$$\zeta_{\nu,x} : \mathcal{O}_\nu \simeq G/G_\nu \to F_{[x]}, \qquad \zeta_{\nu,x}[g] = [\phi_g(x)] \tag{3.6}$$

between the coadjoint orbit $\mathcal{O}_\nu$ and the fiber $F_{[x]}$ is a symplectomorphism.

ii) For any $f_\nu, h_\nu \in C^\infty(M/G_\nu)$ and $[x] \in M/G_\nu$, we have that

$$\{f_\nu, h_\nu\}_{M/G_\nu}[x] = -\nu(\left[\mathrm{d}\bar{f}(\nu), \mathrm{d}\bar{h}(\nu)\right]_{\mathfrak{g}}),$$

where $\bar{f}, \bar{h} \in C^\infty(\mathfrak{g}^*)$ are arbitrary extensions of $f_{\nu|F_{[x]}} \circ \zeta_{\nu,x}$ and $h_{\nu|F_{[x]}} \circ \zeta_{\nu,x}$, respectively.

Proof. i) First of all, we note that the following diagram is commutative

$$\begin{array}{ccc} (J^{V^*\pi})^{-1}(\nu) = M \times \{\nu\} & \xrightarrow{\pi_\nu^{V^*\pi}} & (J^{V^*\pi})^{-1}(\nu)/G_\nu \simeq M/G_\nu \\ {\scriptstyle i_\nu^{V^*\pi}}\downarrow & & \downarrow{\scriptstyle \pi^*_{1,\nu}} \\ V^*\pi \simeq M \times \mathfrak{g}^* & \xrightarrow{\pi_1^*} & M/G \simeq \mathbb{R} \end{array} \tag{3.7}$$

where $i_\nu^{V^*\pi}$ is the inclusion and π_1^* is the map $\pi \circ pr_1$, $pr_1 : M \times \mathfrak{g}^* \to M$ being the first projection.

Now, let $\rho_\nu : M/G_\nu \to \mathbb{R}$ be a real C^∞-function on M/G_ν and $\rho : V^*\pi \simeq M \times \mathfrak{g}^* \to \mathbb{R}$ a G-invariant extension of the real function $\rho_\nu \circ \pi_\nu^{V^*\pi} : (J^{V^*\pi})^{-1}(\nu) \simeq M \times \{\nu\} \to \mathbb{R}$. Then, using Proposition 2.14, we deduce that the restriction of the Hamiltonian vector field $\mathcal{H}_\rho$ to $(J^{V^*\pi})^{-1}(\nu) \simeq M \times \{\nu\}$ is tangent to $(J^{V^*\pi})^{-1}(\nu) \simeq M \times \{\nu\}$ and $(\mathcal{H}_\rho)_{|(J^{V^*\pi})^{-1}(\nu)}$ is $\pi_\nu^{V^*\pi}$-projectable on the Hamiltonian vector field $\mathcal{H}_{\rho_\nu}$ on $(J^{V^*\pi})^{-1}(\nu)/G_\nu \simeq M/G_\nu$.

On the other hand, from (3.3), it follows that the Hamiltonian vector field $\mathcal{H}_\rho$ of the function $\rho \in C^\infty(M \times \mathfrak{g}^*)$ is π_1^*-vertical, that is,

$$\mathcal{H}_\rho(x,\alpha) \in V_{(x,\alpha)}\pi_1^*, \qquad \text{for } (x,\alpha) \in M \times \mathfrak{g}^* \simeq V^*\pi. \tag{3.8}$$

Thus, from (3.7) we obtain that $\mathcal{H}_{\rho_\nu}$ is $\pi^*_{1,\nu}$-vertical. Therefore,

$$(\mathcal{H}_{\rho_\nu})_{|F_{[x]}} \text{ is tangent to } F_{[x]}. \tag{3.9}$$

Next, denote by $S_{[x]}$ the symplectic leaf of the reduced Poisson manifold $(J^{V^*\pi})^{-1}(\nu)/G_\nu \simeq M/G_\nu$ passing through the point $[x]$. Then,

$$\dim S_{[x]} = \dim(M/G_\nu) - 1 = \dim F_{[x]}. \tag{3.10}$$

In addition, it is clear that the map $\zeta_{\nu,x} : \mathcal{O}_\nu \simeq G/G_\nu \to F_{[x]}$ given in (3.6) is a diffeomorphism between the coadjoint orbit $\mathcal{O}_\nu \simeq G/G_\nu$ and the fiber $F_{[x]}$. In particular, using that G is connected, we deduce that $F_{[x]}$ also is connected. Consequently, from (3.9) and (3.10), the fiber $F_{[x]}$ is an open subset of $S_{[x]}$. Thus, since $S_{[x]}$ is connected, we conclude that $S_{[x]} = F_{[x]}$.

Finally, we will see that the diffeomorphism $\zeta_{\nu,x}$ is a symplectomorphism. More precisely, we will prove that the embedding

$$\zeta_{\nu,x} : \mathcal{O}_\nu \simeq G/G_\nu \to (J^{V^*\pi})^{-1}(\nu)/G_\nu \simeq M/G_\nu$$

is a Poisson morphism.

We consider the embedding $\chi_x : G \times \mathfrak{g}^* \simeq T^*G \to M \times \mathfrak{g}^* \simeq V^*\pi$ given by

$$\chi_x(g,\alpha) = (\phi_g(x), \mathrm{Ad}^*_{g^{-1}}\alpha), \qquad \text{for } (g,\alpha) \in G \times \mathfrak{g}^*.$$

Using (1.16) and (3.4), it follows that χ_x is equivariant with respect to the Poisson actions T^*l and $(T^*\phi)^{V^*\pi}$, that is,

$$\chi_x \circ (T^*l)_g = (T^*\phi)_g^{V^*\pi} \circ \chi_x, \qquad \text{for } g \in G. \tag{3.11}$$

Moreover, from (1.50) and (3.5), we deduce that

$$J^{V^*\pi} \circ \chi_x = J^{T^*G}. \tag{3.12}$$

Note that the infinitesimal generator of l associated with $\xi \in \mathfrak{g}$ is the right invariant vector field $\overrightarrow{\xi}$ on G.

Now, we will see that χ_x is a Poisson morphism. It is sufficient to prove that

$$(T_{(g,\alpha)}\chi_x)(\mathcal{H}_{\widehat{\xi}\circ\chi_x}(g,\alpha)) = \mathcal{H}_{\widehat{\xi}}(\chi_x(g,\alpha)), \tag{3.13}$$

$$(T_{(g,\alpha)}\chi_x)(\mathcal{H}_{f\circ pr_1\circ\chi_x}(g,\alpha)) = \mathcal{H}_{f\circ pr_1}(\chi_x(g,\alpha)), \tag{3.14}$$

for $(g,\alpha) \in G \times \mathfrak{g}^*$, $\xi \in \mathfrak{g}$ and $f \in C^\infty(M)$, where $\mathcal{H}_{\widehat{\xi}\circ\chi_x}$ and $\mathcal{H}_{f\circ pr_1\circ\chi_x}$ (respectively, $\mathcal{H}_{\widehat{\xi}}$ and $\mathcal{H}_{f\circ pr_1}$) are the Hamiltonian vector fields in $G \times \mathfrak{g}^* \simeq$

T^*G (respectively, $M \times \mathfrak{g}^* \simeq V^*\pi$) of the real functions $\widehat{\xi} \circ \chi_x$ and $f \circ pr_1 \circ \chi_x$ (respectively, $\widehat{\xi}$ and $f \circ pr_1$).

Note that

$$\widehat{\xi} \circ \chi_x = J_\xi^{T^*G}, \qquad \widehat{\xi} = J_\xi^{V^*\pi} \tag{3.15}$$

and, therefore, since J^{T^*G} and $J^{V^*\pi}$ are momentum maps, we obtain that

$$\mathcal{H}_{\widehat{\xi}\circ\chi_x}(g,\alpha) = \xi_{G\times\mathfrak{g}^*}(g,\alpha), \quad \mathcal{H}_{\widehat{\xi}}(\chi_x(g,\alpha)) = \xi_{M\times\mathfrak{g}^*}(\chi_x(g,\alpha)).$$

This, using (3.11), implies that (3.13) holds.

On the other hand, from (1.15), it follows that

$$\mathcal{H}_{f\circ pr_1\circ\chi_x}(g,\alpha) = \mathcal{H}_{f\circ\phi_x\circ pr_1}(g,\alpha) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}(g, \alpha + t(T_e^*l_g)\mathrm{d}(f\circ\phi_x)(g)).$$

Consequently,

$$\begin{aligned}
&(T_{(g,\alpha)}\chi_x)(\mathcal{H}_{f\circ pr_1\circ\chi_x}(g,\alpha)) \\
&\qquad = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}(\phi_x(g), \mathrm{Ad}^*_{g^{-1}}\alpha + t(\mathrm{Ad}^*_{g^{-1}}\circ T_e^*l_g)\mathrm{d}(f\circ\phi_x)(g)) \\
&\qquad = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}(\phi_x(g), \mathrm{Ad}^*_{g^{-1}}\alpha + t(T_e^*r_g)\mathrm{d}(f\circ\phi_x)(g)) \\
&\qquad = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}(\phi_x(g), \mathrm{Ad}^*_{g^{-1}}\alpha + t(T_e^*\phi_{\phi_g(x)})\mathrm{d}f(\phi_g(x))).
\end{aligned}$$

Thus, if $f' \in C^\infty(M)$ and $\xi' \in \mathfrak{g}$, we have that

$$\begin{aligned}
&\left\{(T_{(g,\alpha)}\chi_x)(\mathcal{H}_{f\circ pr_1\circ\chi_x}(g,\alpha))\right\}(f'\circ pr_1) = 0, \\
&\left\{(T_{(g,\alpha)}\chi_x)(\mathcal{H}_{f\circ pr_1\circ\chi_x}(g,\alpha))\right\}(\widehat{\xi'}) = \xi'_M(\phi_g(x))(f).
\end{aligned}$$

This, using (3.3), proves that (3.14) holds.

Now, from (3.12), we deduce that the restriction of χ_x to $(J^{T^*G})^{-1}(\nu)$ takes values in $(J^{V^*\pi})^{-1}(\nu)$ and $(\chi_x)_{|(J^{T^*G})^{-1}(\nu)} : (J^{T^*G})^{-1}(\nu) \to (J^{V^*\pi})^{-1}(\nu)$ is an embedding. In addition, using (3.11), one has that this embedding passes to the quotient and so, induces an injective immersion

$$\overline{\chi}_x : (J^{T^*G})^{-1}(\nu)/G_\nu \to (J^{V^*\pi})^{-1}(\nu)/G_\nu$$

It is easy to prove that under the identifications $(J^{T^*G})^{-1}(\nu)/G_\nu \simeq G/G_\nu \simeq \mathcal{O}_\nu$ and $(J^{V^*\pi})^{-1}(\nu)/G_\nu \simeq M/G_\nu$, we have that $\overline{\chi}_x$ is just $\zeta_{\nu,x}$. Furthermore, since χ_x is a Poisson morphism, we conclude that $\overline{\chi}_x = \zeta_{\nu,x}$ also is a Poisson morphism.

ii) It follows from (1.6), (1.28) and the fact that $\zeta_{\nu,x}$ is symplectic. In fact, for any $f_\nu, h_\nu \in C^\infty(M/G_\nu)$, we have that

$$\begin{aligned}\{f_\nu, h_\nu\}_{M/G_\nu}[x] &= \left\{f_{\nu|F_{[x]}}, h_{\nu|F_{[x]}}\right\}_{F_{[x]}}[x] \\ &= \left\{f_{\nu|F_{[x]}} \circ \zeta_{\nu,x}, h_{\nu|F_{[x]}} \circ \zeta_{\nu,x}\right\}_{\mathcal{O}_\nu}(\nu) \\ &= -\nu(\left[\mathrm{d}\bar{f}(\nu), \mathrm{d}\bar{h}(\nu)\right]_{\mathfrak{g}}),\end{aligned}$$

where $\{\cdot,\cdot\}_{F_{[x]}}$ is the Poisson bracket on $F_{[x]}$ and $\bar{f}, \bar{h} \in C^\infty(\mathfrak{g}^*)$ are arbitrary extensions of $f_{\nu|F_{[x]}} \circ \zeta_{\nu,x}$ and $h_{\nu|F_{[x]}} \circ \zeta_{\nu,x}$, respectively. $\square$

Remark 3.3. *Under the same hypotheses as in Theorem 3.2, we have that the principal G-bundle $\pi : M \to M/G \simeq \mathbb{R}$ is trivializable (see [32]). So, one may find an isomorphism between this principal G-bundle and the trivial principal G-bundle $pr_2 : G \times \mathbb{R} \to \mathbb{R}$. Using this isomorphism, it is easy to prove that the space of orbits M/G_ν is diffeomorphic to the product manifold $G/G_\nu \times \mathbb{R}$ and, therefore, M/G_ν is a Poisson manifold of corank 1 whose symplectic leaves are isomorphic to the coadjoint orbit $G/G_\nu \simeq \mathcal{O}_\nu$. This result is weaker than Theorem 3.2. In fact, Theorem 3.2 proves that the Poisson structure on M/G_ν and the symplectomorphism between a symplectic leaf of M/G_ν and the coadjoint orbit $G/G_\nu \simeq \mathcal{O}_\nu$ don't depend on the chosen trivialization.*

$\diamond$

3.3 Reduction Theorem for standard symplectic principal $\mathbb{R}$-bundles

In this section we will give sufficient conditions for the reduced symplectic principal $\mathbb{R}$-bundle obtained from a standard principal $\mathbb{R}$-bundle to be again standard. We will obtain a principal $\mathbb{R}$-bundle embedding from the reduced symplectic principal $\mathbb{R}$-bundle at the level $\nu \in \mathfrak{g}^*$ to a certain standard principal $\mathbb{R}$-bundle. This embedding will be a symplectic principal $\mathbb{R}$-bundle isomorphism in the case $\mathfrak{g} = \mathfrak{g}_\nu$.

We will use some well-known results of the cotangent bundle reduction theory (see Section 1.6).

Suppose that a connected Lie group G acts freely and properly on a manifold M. Then G acts symplectically on T^*M by cotangent lift, where T^*M is equipped with the canonical symplectic form Ω_M. We assume that we have a G-invariant submersion $\pi : M \to \mathbb{R}$. Then, the action $T^*\phi$:

$G \times T^*M \to T^*M$ is canonical on the symplectic principal $\mathbb{R}$-bundle $\mu_\pi : (T^*M, \Omega_M) \to V^*\pi$. If we denote by $J^{T^*M} : T^*M \to \mathfrak{g}^*$ and $J^{V^*\pi} : V^*\pi \to \mathfrak{g}^*$ the Ad^*-equivariant momentum maps on T^*M and $V^*\pi$ respectively, we have the corresponding free and proper actions $G_\nu \times (J^{T^*M})^{-1}(\nu) \to (J^{T^*M})^{-1}(\nu)$ and $G_\nu \times (J^{V^*\pi})^{-1}(\nu) \to (J^{V^*\pi})^{-1}(\nu)$ of the isotropy group G_ν of an element $\nu \in \mathfrak{g}^*$.

Using Theorem 2.13, we obtain a reduced symplectic principal $\mathbb{R}$-bundle

$$(\mu_\pi)_\nu : ((T^*M)_\nu, (\Omega_M)_\nu) \to (V^*\pi)_\nu,$$

where $(T^*M)_\nu = (J^{T^*M})^{-1}(\nu)/G_\nu$ and $(V^*\pi)_\nu = (J^{V^*\pi})^{-1}(\nu)/G_\nu$.

On the other hand, since π is G-invariant, there exists a unique surjective submersion $\pi^*_{1,\nu} : M/G_\nu \to \mathbb{R}$ such that

$$\pi^*_{1,\nu} \circ \pi_{M,G_\nu} = \pi, \tag{3.16}$$

where we have denoted by $\pi_{M,G_\nu} : M \to M/G_\nu$ the ν-shape space. Thus, we may consider the corresponding standard symplectic principal $\mathbb{R}$-bundle

$$\mu_{\pi^*_{1,\nu}} : (T^*(M/G_\nu), \Omega_{M/G_\nu}) \to V^*\pi^*_{1,\nu}.$$

Here, Ω_{M/G_ν} denotes the canonical symplectic 2-form on the cotangent bundle $T^*(M/G_\nu)$.

We will prove that, under a suitable hypothesis, the reduced symplectic principal $\mathbb{R}$-bundle $(\mu_\pi)_\nu$ may be embedded into the standard symplectic principal $\mathbb{R}$-bundle $\mu_{\pi^*_{1,\nu}}$, where the total space $T^*(M/G_\nu)$ will be equipped with the canonical symplectic form Ω_{M/G_ν} deformed by a *magnetic term*.

The magnetic term is defined as in Section 1.6.2. We consider the action $\phi_\nu : G_\nu \times M \to M$ deduced from $\phi : G \times M \to M$. Its cotangent lift $T^*\phi_\nu : G_\nu \times T^*M \to T^*M$ has an Ad^*-equivariant momentum map $J^{T^*M}_\nu : T^*M \to \mathfrak{g}^*_\nu$ obtained by restricting J^{T^*M} as in (1.51).

Let $\nu' = \nu_{|\mathfrak{g}_\nu} \in \mathfrak{g}^*_\nu$ be the restriction of ν to $\mathfrak{g}_\nu$. As in Section 1.6.2, we will use the following fact

(MT) There exists a G_ν-invariant 1-form λ_ν on M with values in $(J^{T^*M}_\nu)^{-1}(\nu')$.

Recall that the magnetic term associated with λ_ν is just the 2-form B_{λ_ν} on $T^*(M/G_\nu)$ defined by

$$B_{\lambda_\nu} = \pi^*_{M/G_\nu}\beta_{\lambda_\nu},$$

where $\pi_{M/G_\nu} : T^*(M/G_\nu) \to M/G_\nu$ is the cotangent bundle projection and β_{λ_ν} is the 2-form on M/G_ν characterized by condition

$$\pi^*_{M,G_\nu}\beta_{\lambda_\nu} = \mathrm{d}\lambda_\nu. \tag{3.17}$$

Note that the magnetic term B_{λ_ν} is invariant under the principal action $\widetilde{\psi}$ of $\mu_{\pi^*_{1,\nu}}$. In fact,

$$\widetilde{\psi}_s^* B_{\lambda_\nu} = \widetilde{\psi}_s^*(\pi^*_{M/G_\nu}\beta_{\lambda_\nu}) = \pi^*_{M/G_\nu}\beta_{\lambda_\nu} = B_{\lambda_\nu},$$

where we have used the fact that $\pi_{M/G_\nu} \circ \widetilde{\psi}_s = \pi_{M/G_\nu}$. Moreover, it is well-known (see [1]) that $\Omega_{M/G_\nu} - B_{\lambda_\nu}$ is a symplectic structure on $T^*(M/G_\nu)$. It follows that $\mu_{\pi^*_{1,\nu}} : (T^*(M/G_\nu), \Omega_{M/G_\nu} - B_{\lambda_\nu}) \to V^*\pi^*_{1,\nu}$ is a symplectic principal $\mathbb{R}$-bundle.

The main theorem of this Section is the following one:

Theorem 3.4. *Let $\phi : G \times M \to M$ be a free and proper action of a connected Lie group G on the manifold M and $\pi : M \to \mathbb{R}$ a G-invariant surjective submersion. Let $\nu \in \mathfrak{g}^*$ and $\pi^*_{1,\nu} : M/G_\nu \to \mathbb{R}$ the surjective submersion obtained from π by passing to the quotient. Choose a G_ν-invariant 1-form $\lambda_\nu \in \Omega^1(M)$ with values in $(J_\nu^{T^*M})^{-1}(\nu')$. Then there is a symplectic principal $\mathbb{R}$-bundle embedding*

$$\varphi_{\lambda_\nu} : (T^*M)_\nu \to T^*(M/G_\nu)$$

between the reduced symplectic principal $\mathbb{R}$-bundle

$$(\mu_\pi)_\nu : ((T^*M)_\nu, (\Omega_M)_\nu) \to (V^*\pi)_\nu$$

and the standard symplectic principal $\mathbb{R}$-bundle

$$\mu_{\pi^*_{1,\nu}} : (T^*(M/G_\nu), \Omega_{M/G_\nu} - B_{\lambda_\nu}) \to V^*\pi^*_{1,\nu},$$

with the symplectic structure modified by the magnetic term B_{λ_ν} associated with λ_ν.

Moreover, φ_{λ_ν} is a symplectic principal $\mathbb{R}$-bundle isomorphism if and only if $\mathfrak{g} = \mathfrak{g}_\nu$ (in particular, if $\nu = 0$ or $G = G_\nu$), where $\mathfrak{g}_\nu$ is the Lie algebra of G_ν.

Proof. Using the cotangent bundle reduction theory (see Theorem 1.20), we have that there is a symplectic embedding

$$\varphi_{\lambda_\nu} : (T^*M)_\nu \to T^*(M/G_\nu)$$

which is an isomorphism if and only if $\mathfrak{g} = \mathfrak{g}_\nu$.

Now, using the expression of φ_{λ_ν} given in Section 1.6.2, we will prove that φ_{λ_ν} is a principal $\mathbb{R}$-bundle morphism between $(\mu_\pi)_\nu$ and $\mu_{\pi^*_{1,\nu}}$.

Firstly, we suppose that $G = G_\nu$. In such a case, φ_{λ_ν} is the symplectic isomorphism described as follows. Consider the map

$$\bar{\varphi}_{\lambda_\nu} : (J^{T^*M})^{-1}(\nu) \to T^*(M/G)$$

given by

$$\bar{\varphi}_{\lambda_\nu}(\alpha_x)(T_x\pi_{M,G}(v_x)) = (\alpha_x - \lambda_\nu(x))(v_x)$$

for all $\alpha_x \in (J^{T^*M})^{-1}(\nu) \cap T_x^*M$ and $v_x \in T_xM$. Then, φ_{λ_ν} is the map obtained from $\bar{\varphi}_{\lambda_\nu}$ by passing to the quotient with respect the action

$$\phi : G \times (J^{T^*M})^{-1}(\nu) \to (J^{T^*M})^{-1}(\nu).$$

Now, we prove that $\bar{\varphi}_{\lambda_\nu}$ is equivariant with respect to the $\mathbb{R}$-actions $\psi_\pi : \mathbb{R} \times (J^{T^*M})^{-1}(\nu) \to (J^{T^*M})^{-1}(\nu)$ and $\psi_{\pi^*_{1,\nu}} : \mathbb{R} \times T^*(M/G) \to T^*(M/G)$, that is

$$\bar{\varphi}_{\lambda_\nu} \circ (\psi_\pi)_s = (\psi_{\pi^*_{1,\nu}})_s \circ \bar{\varphi}_{\lambda_\nu}, \qquad \text{for any } s \in \mathbb{R}. \tag{3.18}$$

In fact, for all $\alpha_x \in (J^{T^*M})^{-1}(\nu) \cap T_x^*M$ and $v_x \in T_xM$

$$[(\psi_{\pi^*_{1,\nu}})_s \circ \bar{\varphi}_{\lambda_\nu}](\alpha_x)(T_x\pi_{M,G}(v_x)) = (\alpha_x - \lambda_\nu(x))(v_x) + s\,(\pi^*_{M,G}(\widetilde{\eta}))(x)(v_x)$$

where $\widetilde{\eta} = (\pi^*_{1,\nu})^*(dt)$. Therefore, from (3.16), we deduce (3.18).

If $\pi_\nu : (J^{T^*M})^{-1}(\nu) \to (J^{T^*M})^{-1}(\nu)/G_\nu$ denotes the quotient map, from (3.18) and since $\varphi_{\lambda_\nu} \circ \pi_\nu = \bar{\varphi}_{\lambda_\nu}$ and π_ν is equivariant with respect to the principal $\mathbb{R}$-actions, we have that

$$\varphi_{\lambda_\nu} \circ ((\psi_\pi)_\nu)_s \circ \pi_\nu = \varphi_{\lambda_\nu} \circ \pi_\nu \circ (\psi_\pi)_s = (\psi_{\pi^*_{1,\nu}})_s \circ \varphi_{\lambda_\nu} \circ \pi_\nu,$$

$(\psi_\pi)_\nu : \mathbb{R} \times (T^*M)_\nu \to (T^*M)_\nu$ being the $\mathbb{R}$-action on the reduced space $(T^*M)_\nu$ induced by ψ_π. Thus, using the fact that π_ν is surjective, we conclude that φ_{λ_ν} is a symplectic principal $\mathbb{R}$-bundle morphism in the case $G = G_\nu$.

Finally, suppose that ν is an arbitrary element of $\mathfrak{g}^*$. In such a case, φ_{λ_ν} is the composition of a symplectic embedding ι with a symplectomorphism $\widetilde{\varphi}_{\lambda_\nu}$ (see Section 1.6.2).

We consider the action $\phi_\nu : G_\nu \times M \to M$ induced by ϕ. Its cotangent lift $T^*\phi_\nu : G_\nu \times T^*M \to T^*M$ is a symplectic action which admits an Ad^*-equivariant momentum map $J_\nu^{T^*M} : T^*M \to \mathfrak{g}_\nu^*$ given by (1.51). If $\nu' = \nu_{|\mathfrak{g}_\nu} \in \mathfrak{g}_\nu^*$ then ν' is a fixed point of the coadjoint action of G_ν, i.e. $(G_\nu)_{\nu'} = G_\nu$.

Recall that the map ι is obtained from the G_ν-invariant embedding $\bar{\iota}$ given in (1.52) by passing to the quotient (see Section 1.6.2). Note that $\bar{\iota}$ is equivariant with respect to the $\mathbb{R}$-actions $\psi_\pi : \mathbb{R} \times (J^{T^*M})^{-1}(\nu) \to (J^{T^*M})^{-1}(\nu)$

and $\psi_\pi : \mathbb{R} \times (J_\nu^{T^*M})^{-1}(\nu') \to (J_\nu^{T^*M})^{-1}(\nu')$. Thus, ι is equivariant with respect to the reduced $\mathbb{R}$-actions

$$(\psi_\pi)_\nu : \mathbb{R} \times (J^{T^*M})^{-1}(\nu)/G_\nu \to (J^{T^*M})^{-1}(\nu)/G_\nu$$

and

$$(\psi_\pi)_{\nu'} : \mathbb{R} \times (J_\nu^{T^*M})^{-1}(\nu')/G_\nu \to (J_\nu^{T^*M})^{-1}(\nu')/G_\nu.$$

On the other hand, $(J^{T^*M})^{-1}(\nu)/G_\nu$ (respectively, $(J_\nu^{T^*M})^{-1}(\nu')/G_\nu$) is the total space of the reduced symplectic principal $\mathbb{R}$-bundle $(\mu_\pi)_\nu$ (respectively, $(\mu_\pi)_{\nu'}$) obtained from μ_π using the canonical action of G (respectively, G_ν) on T^*M. Consequently, ι is a symplectic principal $\mathbb{R}$-bundle embedding.

Now, using that $(G_\nu)_{\nu'} = G_\nu$ and the first part of the proof, we have the symplectic principal $\mathbb{R}$-bundle isomorphism

$$\widetilde{\varphi}_{\lambda_\nu} : (T^*M)_{\nu'} \to T^*(M/G_\nu)$$

between the reduced symplectic principal $\mathbb{R}$-bundle $(\mu_\pi)_{\nu'} : ((T^*M)_{\nu'}, (\Omega_M)_{\nu'}) \to (V^*\pi)_{\nu'}$ and $\mu_{\pi^*_{1,\nu}} : (T^*(M/G_\nu), \Omega_{M/G_\nu} - B_{\lambda_\nu}) \to V^*\pi^*_{1,\nu}$.

Since φ_{λ_ν} is the composition of ι with $\widetilde{\varphi}_{\lambda_\nu}$, it is a symplectic principal $\mathbb{R}$-bundle embedding. □

3.4 Description of the Poisson structure on the reduced symplectic principal $\mathbb{R}$-bundle

In the previous section we have shown that, under suitable hypotheses, the symplectic principal $\mathbb{R}$-bundle obtained by reducing a standard principal $\mathbb{R}$-bundle may be embedded in a new standard symplectic principal $\mathbb{R}$-bundle

$$\mu_{\pi^*_{1,\nu}} : (T^*(M/G_\nu), \Omega_{M/G_\nu} - B_{\lambda_\nu}) \to V^*\pi^*_{1,\nu}$$

whose canonical symplectic 2-form Ω_{M/G_ν} on the total space is deformed by the magnetic term.

In order to describe the Poisson structure on the total and base space of this principal $\mathbb{R}$-bundle, we will describe the Poisson structures on the total space and on the base space of a standard symplectic principal $\mathbb{R}$-bundle with the canonical symplectic 2-form deformed by a basic closed 2-form.

Let $\pi : P \to \mathbb{R}$ be a surjective submersion with total space a manifold P of dimension $n+1$ and β a closed 2-form on P. Consider the closed basic 2-form $B = \pi_P^*\beta$ on T^*P, where $\pi_P : T^*P \to P$ is the canonical projection. An easy computation shows that B is invariant with respect to the $\mathbb{R}$-principal action of the standard symplectic principal $\mathbb{R}$-bundle $\mu_\pi : T^*P \to V^*\pi$. Thus, if Ω_P

is the canonical symplectic form on T^*P, $\mu_\pi : (T^*P, \Omega_P - B) \to V^*\pi$ is a symplectic principal $\mathbb{R}$-bundle.

Denote by

- Λ_{T^*P} and $\Lambda^B_{T^*P}$ the Poisson 2-vectors on T^*P induced by the symplectic 2-forms Ω_P and $\Omega_P - B$, respectively
- $\Lambda_{V^*\pi}$ and $\Lambda^B_{V^*\pi}$ the Poisson 2-vectors on $V^*\pi$ induced on the base space of the symplectic principal $\mathbb{R}$-bundles $\mu_\pi : (T^*P, \Omega_P) \to V^*\pi$ and $\mu_\pi : (T^*P, \Omega_P - B) \to V^*\pi$, respectively.

Note that the Poisson 2-vector $\Lambda_{V^*\pi}$ is given as in (2.1) and it is linear.

On the other hand, if $\tau_E : E \to Q$ is a vector bundle over Q and γ is a section of the vector bundle $\wedge^p E \to Q$, then we will denote by γ^v the vertical lift of γ which is a p-vector on E. In fact, if (q^i) are local coordinates on an open subset U of Q, $\{e_\alpha\}$ is a local basis of $\Gamma(E)$ on U and

$$\gamma = \gamma^{\alpha_1 \dots \alpha_p} e_{\alpha_1} \wedge \dots \wedge e_{\alpha_p} \quad \text{on } U$$

then

$$\gamma^v = \gamma^{\alpha_1 \dots \alpha_p} \frac{\partial}{\partial y^{\alpha_1}} \wedge \dots \wedge \frac{\partial}{\partial y^{\alpha_p}}$$

where (q^i, y^α) are the corresponding local coordinates on E.

Proposition 3.5. *Let $\pi : P \to \mathbb{R}$ be a surjective submersion and β a closed 2-form on P. Then*

1) $\Lambda^B_{T^*P} = \Lambda_{T^*P} + \beta^v$, where β^v is the vertical lift to T^*P of β;

2) $\Lambda^B_{V^*\pi} = \Lambda_{V^*\pi} + \bar{\beta}^v$, where $\bar{\beta}^v$ is the vertical lift to $V^*\pi$ of the 2-section $\bar{\beta} = i^*\beta$ of $V^*\pi$ with $i : V\pi \to TP$ the canonical inclusion.

Proof. If $\{\cdot,\cdot\}_{T^*P}$, $\{\cdot,\cdot\}^B_{T^*P}$ are the Poisson brackets on T^*P induced by the symplectic forms Ω_P, $\Omega_P - B$, respectively, then, one may easily prove that (see [42])

$$\{F, F'\}^B_{T^*P} = \{F, F'\}_{T^*P} + B(\mathcal{H}_F, \mathcal{H}_{F'}), \tag{3.19}$$

for any $F, F' \in C^\infty(T^*P)$, where $\mathcal{H}_F, \mathcal{H}_{F'} \in \mathfrak{X}(T^*P)$ are the Hamiltonian vector fields of F, F' with respect to the symplectic structure Ω_P.

Alternatively (3.19) may be written as

$$\{F, F'\}^B_{T^*P} = \{F, F'\}_{T^*P} + \beta^v(\mathrm{d}F, \mathrm{d}F'). \tag{3.20}$$

In fact, it is sufficient to prove that if F and F' are linear or basic functions on T^*P then $\beta^v(\mathrm{d}F, \mathrm{d}F') = B(\mathcal{H}_F, \mathcal{H}_{F'})$. Indeed, we have

$$\begin{aligned}
&B(\mathcal{H}_{\widehat{Y}}, \mathcal{H}_{\widehat{Y'}}) = \beta(Y, Y') \circ \pi_P = \beta^v(\mathrm{d}\widehat{Y}, \mathrm{d}\widehat{Y'}),\\
&B(\mathcal{H}_{\bar{F}\circ\pi_P}, \mathcal{H}_{\widehat{Y}}) = 0 = \beta^v(\mathrm{d}(\bar{F}\circ\pi_P), \mathrm{d}\widehat{Y}),\\
&B(\mathcal{H}_{\bar{F}\circ\pi_P}, \mathcal{H}_{\bar{F}'\circ\pi_P}) = 0 = \beta^v(\mathrm{d}(\bar{F}\circ\pi_P), \mathrm{d}(\bar{F}'\circ\pi_P)),
\end{aligned}$$

for all $\bar{F}, \bar{F}' \in C^\infty(P)$ and $Y, Y' \in \mathfrak{X}(P)$. Therefore, from (3.20), we deduce that the Poisson $\Lambda^B_{T^*P}$ and Λ_{T^*P} are related as follows

$$\Lambda^B_{T^*P} = \Lambda_{T^*P} + \beta^v.$$

On the other hand, the section $\bar{\beta}$ of the vector bundle $\wedge^2 V^*\pi \to P$ is defined by

$$\bar{\beta}(x) = \beta(x)_{|V_x\pi \times V_x\pi}, \qquad \text{for any } x \in P.$$

If $\{\cdot,\cdot\}^B_{V^*\pi}$ and $\{\cdot,\cdot\}_{V^*\pi}$ denote the Poisson brackets on $V^*\pi$ induced by $\Lambda^B_{V^*\pi}$ and $\Lambda_{V^*\pi}$, respectively, from (3.20) and Proposition 2.6, we have that

$$\{f, f'\}^B_{V^*\pi} \circ \mu_\pi = \{f, f'\}_{V^*\pi} \circ \mu_\pi + \beta^v(\mu_\pi^*(\mathrm{d}f), \mu_\pi^*(\mathrm{d}f')), \tag{3.21}$$

for $f, f' \in C^\infty(V^*\pi)$.

Moreover, one may easily prove that

$$\beta^v(\mu_\pi^*(\mathrm{d}f), \mu_\pi^*(\mathrm{d}f')) = \bar{\beta}^v(\mathrm{d}f, \mathrm{d}f') \circ \mu_\pi.$$

Thus,

$$\Lambda^B_{V^*\pi} = \Lambda_{V^*\pi} + \bar{\beta}^v.$$

□

If $(q^0 = t, q^1, \ldots, q^n)$ are coordinates on P adapted to the surjective submersion $\pi : P \to \mathbb{R}$ and $\beta = \frac{1}{2}\beta_{ij}\mathrm{d}q^i \wedge \mathrm{d}q^j$, with $\beta_{ij} = -\beta_{ji}$, is the local expression of $\beta \in \Omega^2(P)$, then the induced Poisson structure $\{\cdot,\cdot\}^B_{T^*P}$ is characterized by

$$\begin{aligned}
\{q^i, q^j\}^B_{T^*P} &= 0\\
\{q^i, p_j\}^B_{T^*P} &= \delta^i_j,\\
\{p_i, p_j\}^B_{T^*P} &= \beta_{ij},
\end{aligned}$$

for $i, j = 0, 1, \ldots, n$.

The base space $V^*\pi$ of the symplectic principal $\mathbb{R}$-bundle $\mu_\pi : (T^*P, \Omega_P - B) \to V^*\pi$ has coordinates $(t = q^0, q^i, p_i)$ (with $i = 1, \dots, n$) and the local expression of the corresponding Poisson bracket is

$$\begin{aligned} \left\{t, q^i\right\}^B_{V^*\pi} &= \{t, p_j\}^B_{V^*\pi} = 0, \\ \left\{q^i, q^j\right\}^B_{V^*\pi} &= 0 \\ \left\{q^i, p_j\right\}^B_{V^*\pi} &= \delta^i_j, \\ \{p_i, p_j\}^B_{V^*\pi} &= \beta_{ij}, \end{aligned}$$

for $i, j = 1, \dots, n$.

Now, let $\phi : G \times M \to M$ be a free and proper action of a connected Lie group G on M and $\pi : M \to \mathbb{R}$ a G-invariant surjective submersion. Let $\nu \in \mathfrak{g}^*$ and $\pi^*_{1,\nu} : M/G_\nu \to \mathbb{R}$ the surjective submersion obtained from π by passing to the quotient. Choose a G_ν-invariant 1-form $\lambda_\nu \in \Omega^1(M)$ with values in $(J^{T^*M}_\nu)^{-1}(\nu')$.

Then, from Theorem 3.4, we deduce that there is a symplectic principal $\mathbb{R}$-bundle embedding $\varphi_{\lambda_\nu} : (T^*M)_\nu \to T^*(M/G_\nu)$ between the reduced symplectic principal $\mathbb{R}$-bundle $(\mu_\pi)_\nu : ((T^*M)_\nu, (\Omega_M)_\nu) \to (V^*\pi)_\nu$ and the standard symplectic principal $\mathbb{R}$-bundle $\mu_{\pi^*_{1,\nu}} : (T^*(M/G_\nu), \Omega_{M/G_\nu} - B_{\lambda_\nu}) \to V^*\pi^*_{1,\nu}$, with the symplectic structure modified by the magnetic term $B_{\lambda_\nu} \in \Omega^2(T^*(M/G_\nu))$ associated with λ_ν.

We recall that if $\pi_{M,G_\nu} : M \to M/G_\nu$ is the canonical projection and β_{λ_ν} is the closed 2-form on M/G_ν satisfying $\pi^*_{M,G_\nu}\beta_{\lambda_\nu} = d\lambda_\nu$ then B_{λ_ν} is $\pi^*_{M/G_\nu}\beta_{\lambda_\nu}$.

Thus, using Proposition 3.5, it follows that the Poisson structures on $T^*(M/G_\nu)$ and $V^*\pi^*_{1,\nu}$ are

$$\Lambda_{T^*(M/G_\nu)} + \beta^v_{\lambda_\nu} \text{ and } \Lambda_{V^*\pi^*_{1,\nu}} + \bar{\beta}^v_{\lambda_\nu},$$

respectively.

On the other hand, if $\varphi^V_{\lambda_\nu} : (V^*\pi)_\nu \to V^*\pi^*_{1,\nu}$ is the corresponding embedding between the base spaces of the principal $\mathbb{R}$-bundles, $[\alpha_x] \in (T^*M)_\nu$ and $[\bar{\alpha}_x] \in (V^*\pi)_\nu$, then, from (2.9) and (2.11), we have that

$$\begin{aligned} T^*_{\varphi_{\lambda_\nu}[\alpha_x]}(T^*(M/G_\nu)) &= \left(T_{[\alpha_x]}(T^*M)_\nu\right)^o \oplus \left(\sharp_{\Lambda_{T^*(M/G_\nu)}+\beta^v_{\lambda_\nu}}(T_{[\alpha_x]}(T^*M)_\nu)^o\right)^o, \\ T^*_{\varphi^V_{\lambda_\nu}[\bar{\alpha}_x]}(V^*\pi^*_{1,\nu}) &= \left(T_{[\bar{\alpha}_x]}(V^*\pi)_\nu\right)^o \oplus \left(\sharp_{\Lambda_{V^*\pi^*_{1,\nu}}+\bar{\beta}^v_{\lambda_\nu}}(T_{[\bar{\alpha}_x]}(V^*\pi)_\nu)^o\right)^o, \end{aligned}$$

and the corresponding projectors

$$\begin{aligned} \widetilde{P}_{[\alpha_x]} &: T^*_{\varphi_{\lambda_\nu}[\alpha_x]}(T^*(M/G_\nu)) \to \left(\sharp_{\Lambda_{T^*(M/G_\nu)}+\beta^v_{\lambda_\nu}}(T_{[\alpha_x]}(T^*M)_\nu)^o\right)^o, \\ \widetilde{p}^*_{[\bar{\alpha}_x]} &: T^*_{\varphi^V_{\lambda_\nu}[\bar{\alpha}_x]}(V^*\pi^*_{1,\nu}) \to \left(\sharp_{\Lambda_{V^*\pi^*_{1,\nu}}+\bar{\beta}^v_{\lambda_\nu}}(T_{[\bar{\alpha}_x]}(V^*\pi)_\nu)^o\right)^o. \end{aligned}$$

Moreover, using Proposition 2.10, we conclude that

Theorem 3.6. *Under the same hypotheses as in Theorem 3.4, the reduced Poisson structures Λ_ν and $\bar{\Lambda}_\nu$ on $(T^*M)_\nu$ and $(V^*\pi)_\nu$, respectively, are given by*

$$\Lambda_\nu([\alpha_x])\big(\varphi^*_{\lambda_\nu}\alpha'_1, \varphi^*_{\lambda_\nu}\alpha'_2\big) = (\Lambda_{T^*(M/G_\nu)} + \beta^v_{\lambda_\nu})(\varphi_{\lambda_\nu}[\alpha_x])\big(\widetilde{P}_{[\alpha_x]}\alpha'_1, \widetilde{P}_{[\alpha_x]}\alpha'_2\big)$$

and

$$\bar{\Lambda}_\nu([\bar{\alpha}_x])\big((\varphi^V_{\lambda_\nu})^*\sigma'_1, (\varphi^V_{\lambda_\nu})^*\sigma'_2\big) = (\Lambda_{V^*\pi^*_{1,\nu}} + \bar{\beta}^v_{\lambda_\nu})(\varphi^V_{\lambda_\nu}[\bar{\alpha}_x])\big(\widetilde{p}_{[\bar{\alpha}_x]}\sigma'_1, \widetilde{p}_{[\bar{\alpha}_x]}\sigma'_2\big).$$

for all $\alpha'_1, \alpha'_2 \in T^*_{\varphi_{\lambda_\nu}[\alpha_x]}(T^*(M/G_\nu))$, $\sigma'_1, \sigma'_2 \in T^*_{\varphi^V_{\lambda_\nu}[\bar{\alpha}_x]}(V^*\pi^*_{1,\nu})$, $[\alpha_x] \in (T^*M)_\nu$ and $[\bar{\alpha}_x] \in (V^*\pi)_\nu$.

3.5 Bundle version of the standard reduction

In this section we will give an other description of the reduced symplectic principal bundle $\mathbb{R}$-bundle. In particular, we will obtain an interpretation of the magnetic term as the curvature of a suitable principal connection. We will consider the following set up: let $\pi : M \to \mathbb{R}$ be a surjective submersion, Ω_M the canonical symplectic structure on T^*M and $\mu_\pi : (T^*M, \Omega_M) \to V^*\pi$ the standard symplectic principal $\mathbb{R}$-bundle associated with π. We will suppose that a connected Lie group G acts freely and properly on M. Thus, we have a principal G-action $\phi : G \times M \to M$ with corresponding projection $\pi_{M,G} : M \to M/G$. Moreover, we will suppose that a principal connection $\lambda : TM \to \mathfrak{g}$ is given.

As we already know, the cotangent lift action $T^*\phi : G \times T^*M \to T^*M$ is symplectic and admits an Ad^*-equivariant momentum map $J^{T^*M} : T^*M \to \mathfrak{g}^*$ given by (1.50). If we apply the orbit version of the reduction procedure to the symplectic manifold (T^*M, Ω_M) (see Section 1.5.2), we obtain that the reduced symplectic manifold is $((T^*M)_\mathcal{O}, (\Omega_M)_\mathcal{O})$, $\mathcal{O}$ being the coadjoint orbit of an element $\nu \in \mathfrak{g}^*$ and $(T^*M)_\mathcal{O} = (J^{T^*M})^{-1}(\mathcal{O})/G$. Moreover, $((T^*M)_\mathcal{O}, (\Omega_M)_\mathcal{O})$ is symplectomorphic to $(T^*(M/G) \times_{M/G} \widetilde{\mathcal{O}}, \Omega_{\mathrm{red}})$, where $\widetilde{\mathcal{O}} = (M \times \mathcal{O})/G$ and Ω_{red} is described in Section 1.6.3.

A symplectomorphism $\Theta : (J^{T^*M})^{-1}(\mathcal{O})/G \to T^*(M/G) \times_{M/G} \widetilde{\mathcal{O}}$ is explicitely given by

$$\pi_\mathcal{O}(\alpha_x) \mapsto (\mathrm{Hor}^*_x\alpha_x, [x, J^{T^*M}(\alpha_x)]), \tag{3.22}$$

where $\mathrm{Hor}^*_x : T^*M \to T_{[x]}(M/G)$ is the dual map of the horizontal lift at x (see Section 1.6.1).

On the other hand, as we already know, if $\pi : M \to \mathbb{R}$ is G-invariant, i.e. $\pi \circ \phi_g = \pi$ for any $\mathfrak{g} \in G$, the cotangent lift action $T^*\phi : G \times T^*M \to$

T^*M is a canonical action on the standard symplectic principal $\mathbb{R}$-bundle. Moreover, a surjective submersion $\pi_1^* : M/G \to \mathbb{R}$ may be obtained from π by passing to the quotient. The map π_1^* is characterized by

$$\pi = \pi_1^* \circ \pi_{M,G}. \tag{3.23}$$

The next result, whose proof is direct, shows that, under this hypothesis, $T^*(M/G) \times \widetilde{\mathcal{O}}$ is the total space of a symplectic principal $\mathbb{R}$-bundle.

Proposition 3.7. *Let $\phi : G \times M \to M$ a free and proper action of a connected Lie group on a manifold M. Suppose that $\pi : M \to \mathbb{R}$ is a G-invariant surjective submersion. Then, the surjective submersion*

$$\bar{\mu}_{\mathcal{O}} : T^*(M/G) \times_{M/G} \widetilde{\mathcal{O}} \to V^*\pi_1^* \times_{M/G} \widetilde{\mathcal{O}}, \qquad (\alpha_{[x]}, [x,\kappa]) \mapsto (\alpha_{x|V_{[x]}\pi_1^*}, [x,\kappa])$$

defines a symplectic principal $\mathbb{R}$-bundle. Its principal $\mathbb{R}$-action

$$\bar{\psi}_{\mathcal{O}} : \mathbb{R} \times T^*(M/G) \times_{M/G} \widetilde{\mathcal{O}} \to T^*(M/G) \times_{M/G} \widetilde{\mathcal{O}},$$

is given by

$$(s, (\alpha_{[x]}, [x,\kappa]) \mapsto (\alpha_{[x]} + s((\pi_1^*)^*\mathrm{d}t)([x]), [x,\kappa]) \tag{3.24}$$

for any $s \in \mathbb{R}$, $x \in M$, $\alpha_{[x]} \in T^*_{[x]}(M/G)$, $\kappa \in \mathfrak{g}^*$. Here, $[x]$ denotes the orbit of x with respect to the action ϕ.

The symplectic principal $\mathbb{R}$-bundle $\bar{\mu}_{\mathcal{O}}$ in the previous proposition describes the exact nature of the reduced symplectic principal $\mathbb{R}$-bundle $\mu_{\mathcal{O}} : (T^*M)_{\mathcal{O}} \to (V^*\pi)_{\mathcal{O}}$, as the following result shows.

Theorem 3.8. *If the hypotheses of Proposition 3.7 hold and a connection $\lambda : TM \to \mathfrak{g}$ on the principal bundle $\pi : M \to M/G$ is fixed, the map Θ given in (3.22) is a symplectic principal $\mathbb{R}$-bundle isomorphism between*

$$\mu_{\mathcal{O}} : ((T^*M)_{\mathcal{O}}, (\Omega_M)_{\mathcal{O}}) \to (V^*\pi)_{\mathcal{O}}$$

and

$$\bar{\mu}_{\mathcal{O}} : T^*(M/G) \times_{M/G} \widetilde{\mathcal{O}} \to V^*\pi_1^* \times_{M/G} \widetilde{\mathcal{O}}.$$

Proof. Since Θ is a symplectomorphism, it's sufficient to prove that the map Θ is equivariant with respect to the principal $\mathbb{R}$-actions $\psi_{\mathcal{O}}$ and $\bar{\psi}_{\mathcal{O}}$. Using (1.46) and (3.23), one may easily show that

$$T_x\pi \circ \mathrm{Hor}_x = T_{[x]}\pi_1^*, \tag{3.25}$$

where $\mathrm{Hor}_x : T_{[x]}(M/G) \to T_xM$ is the horizontal lift at x with respect to the fixed connection $\lambda : TM \to \mathfrak{g}$. As a consequence, we obtain that for any $x \in M$

$$\mathrm{Hor}_x^*((\pi^*(\mathrm{d}t))(x)) = ((\pi_1^*)^*\mathrm{d}t)([x]).$$

Moreover, using (2.15) (which follows from the canonicity of the action $T^*\phi$) and the fact that $(\pi^*(\mathrm{d}t))(x) = \psi_1(0_x)$, we have that

$$J^{T^*M}((\pi^*(\mathrm{d}t))(x)) = 0.$$

Thus, we get

$$\begin{aligned}
\Theta((\psi_{\mathcal{O}})_s(\pi_{\mathcal{O}}(\alpha_x))) &= \Theta(\pi_{\mathcal{O}}(\alpha_x + s(\pi^*(\mathrm{d}t))(x))) \\
&= \big(\mathrm{Hor}_x^*(\alpha_x + s(\pi^*(\mathrm{d}t))(x))), [x, J^{T^*M}(\alpha_x + s(\pi^*(\mathrm{d}t))(x))]\big) \\
&= \big(\mathrm{Hor}_x^*\alpha_x + s((\pi_1^*)^*(\mathrm{d}t))([x]), [x, J^{T^*M}(\alpha_x)]\big) \\
&= (\bar{\psi}_{\mathcal{O}})_s \big(\mathrm{Hor}_x^*\alpha_x, [x, J^{T^*M}(\alpha_x)]\big) \\
&= (\bar{\psi}_{\mathcal{O}})_s(\Theta(\pi_{\mathcal{O}}(\alpha_x)))
\end{aligned}$$

for any $\alpha_x \in (J^{T^*M})^{-1}(\mathcal{O})$. □

As we know, the base spaces $(V^*\pi)_{\mathcal{O}}$ and $V^*\pi_1^* \times_{M/G} \widetilde{\mathcal{O}}$ of the previous symplectic principal $\mathbb{R}$-bundles are Poisson manifolds. Moreover, if we denote by $J^{V^*\pi} : V^*\pi \to \mathfrak{g}^*$ the Poisson momentum map, then $(V^*\pi)_{\mathcal{O}}$ is just the quotient space $(J^{V^*\pi})^{-1}(\mathcal{O})/G$ (see Section 1.5.2).

On the other hand, if $x \in M$, from (3.25), it follows that there exists a monomorphism of real vector spaces $\overline{Hor}_x : V_{[x]}\pi_1^* \to V_x\pi$ such that the following diagram is commutative

$$\begin{tikzcd}
T_{[x]}(M/G) \arrow[r, "Hor_x"] & T_xM \\
V_{[x]}\pi_1^* \arrow[u, hook] \arrow[r, "\overline{Hor}_x"] & V_x\pi \arrow[u, hook]
\end{tikzcd}$$

Now, if $\pi_{\mathcal{O}}^{V^*\pi} : (J^{V^*\pi})^{-1}(\mathcal{O}) \to (V^*\pi)_{\mathcal{O}} = (J^{V^*\pi})^{-1}(\mathcal{O})/G$ is the canonical projection, we may consider the map

$$\Theta^{V^*\pi} : (V^*\pi)_{\mathcal{O}} = (J^{V^*\pi})^{-1}(\mathcal{O})/G \to V^*\pi_1^* \times_{M/G} \widetilde{\mathcal{O}}$$

given by

$$\Theta^{V^*\pi}(\pi_{\mathcal{O}}^{V^*\pi}(\bar{\alpha}_x)) = ((\overline{Hor}_x)^*(\bar{\alpha}_x), [(x, J^{V^*\pi}(\bar{\alpha}_x))]), \tag{3.26}$$

for $\bar{\alpha}_x \in (J^{V^*\pi})^{-1}(\mathcal{O})$.

Using Proposition 2.10, one may prove that $\Theta^{V^*\pi}$ is just the Poisson diffeomorphism induced by the symplectic principal $\mathbb{R}$-bundle isomorphism $\Theta : (T^*M)_{\mathcal{O}} = (J^{T^*M})^{-1}(\mathcal{O})/G \to T^*(M/G) \times_{M/G} \widetilde{\mathcal{O}}$.

Thus, we conclude that

Corollary 3.9. *Under the same hypotheses as in Proposition 3.7, the map* $\Theta^{V^*\pi} : (V^*\pi)_{\mathcal{O}} = (J^{V^*\pi})^{-1}(\mathcal{O})/G \to V^*\pi_1^* \times_{M/G} \widetilde{\mathcal{O}}$ *given by (3.26) is a Poisson diffeomorphism.*

We will apply the previous results to two extreme cases:

- the isotropy group G_ν coincides with the whole Lie group G (for example, if G is abelian);
- the surjective submersion $\pi : M \to \mathbb{R}$ is the bundle projection of the principal G-bundle $\pi_{M,G} : M \to M/G$.

3.5.1 The case $G = G_\nu$

If $G = G_\nu$, the fibers of $\widetilde{\mathcal{O}}$ collapse and we are left with just the symplectic principal $\mathbb{R}$-bundle

$$\bar{\mu}_{\mathcal{O}} \colon (T^*(M/G), \Omega_{\mathrm{red}}) \to V^*\pi_1^*. \tag{3.27}$$

From equations (1.55) and (1.56) (see also [43]) since all brackets vanish, the reduced symplectic form Ω_{red} on $T^*(M/G)$ is given by

$$\Omega_{\mathrm{red}} = \Omega_{M/G} - \pi_{M/G}^*\beta,$$

where $\pi_{M/G} : T^*(M/G) \to M/G$ is the cotangent bundle projection and β is the 2-form on $T^*(M/G)$ characterized by

$$\pi_{M,G}^*\beta = \langle \nu, \mathrm{curv} \rangle.$$

Here $\pi_{M,G} : M \to M/G$ is the principal projection and $\langle \nu, \mathrm{curv} \rangle$ is the 2-form on M given by

$$\langle \nu, \mathrm{curv} \rangle(x)(v_x, w_x) = \langle \nu, \mathrm{curv}_x(v_x, w_x) \rangle, \quad \text{for any } v_x, w_x \in T_xM,$$

$\mathrm{curv} : TM \times_M TM \to \mathfrak{g}$ being the curvature 2-form of the connection λ.

Using the results in Section 3.4 (see Proposition 3.5), we have that the Poisson bivector on the base space $V^*\pi_1^*$ of the symplectic principal $\mathbb{R}$-bundle $\bar{\mu}_{\mathcal{O}}$ is

$$\Lambda_{V^*\pi_1^*} + \bar{\beta}^v,$$

where $\Lambda_{V^*\pi_1^*}$ is the Poisson bivector on $V^*\pi_1^*$ induced on the base space of the standard symplectic principal $\mathbb{R}$-bundle $\mu_{\pi_1^*} : (T^*(M/G), \Omega_{M/G}) \to V^*\pi_1^*$ and $\bar{\beta}^v$ is the vertical lift to $V^*\pi_1^*$ of the 2-section $\bar{\beta} = i_1^*\beta$ of $V\pi_1^*$ with $i_1 : V\pi_1^* \to T(M/G)$ the canonical inclusion.

Notice that the 2-section $\bar{\beta}$ of $V\pi_1^*$ may be obtained via an alternative way: consider the 2-section $i^*\langle\nu, \mathrm{curv}\rangle$ of $V\pi$, where $i : V\pi \to TM$ the canonical inclusion. Using (3.23), one may easily construct the following commutative diagram

$$\begin{tikzcd} V\pi \arrow[r, hook] \arrow[d, "T\pi_{M,G|V\pi}"'] & TM \arrow[d, "T\pi_{M,G}"] \\ V\pi_1^* \arrow[r, hook] & T(M/G) \end{tikzcd}$$

Then, one may easily prove that $\bar{\beta}$ may be expressed in terms of the curvature of λ as

$$\bar{\beta}[x](\bar{v}_{[x]}, \bar{w}_{[x]}) = (i^*\langle\nu, \mathrm{curv}\rangle)(x)(v_x, w_x) \quad \text{for any } \bar{v}_{[x]}, \bar{w}_{[x]} \in V_{[x]}\pi_1^*,$$

where $v_x, w_x \in V_x\pi$ are such that $T_x\pi_{M,G}(v_x) = \bar{v}_{[x]}$ and $T_x\pi_{M,G}(w_x) = \bar{w}_{[x]}$.

3.5.2 The case $\pi : M \to \mathbb{R}$ principal G-bundle

Now, we study the case in which the projection $\pi : M \to \mathbb{R}$ corresponds just with a principal bundle $\pi = \pi_{M,G} : M \to M/G \simeq \mathbb{R}$, where the base space is the real line. As usually, the principal action of the structure group G on M will be denoted by $\phi : G \times M \to M$

In such a case, we have already seen (see Section 3.2) that, if we apply the reduction procedure to the standard symplectic principal $\mathbb{R}$-bundle $\mu_\pi : (T^*M, \Omega) \to V^*\pi$ with respect to the canonical action $T^*\phi$ at a level $\nu \in \mathfrak{g}^*$, we obtain a reduced symplectic principal $\mathbb{R}$-bundle whose base space is canonically diffeomorphic to M/G_ν, where G_ν is the isotropy group of ν.

On the other hand, when we quotient the projection $\pi\colon M \to \mathbb{R}$ with respect to the action of G, we obtain the identity map on $\mathbb{R}$. Thus, from the previous section, the base space of the orbit reduced symplectic principal $\mathbb{R}$-bundle becames just $\widetilde{\mathcal{O}} = (M \times \mathcal{O})/G$, where $\mathcal{O}$ is the coadjoint orbit of ν.

As a consequence, we deduce the following result in which we give an other description of the canonical Poisson structure on M/G_ν.

Theorem 3.10. *Let $\phi : G \times M \to M$ be a (free and proper) principal action of a Lie group G on M such that the quotient space is the real line. Denote by $\pi : M \to M/G \simeq \mathbb{R}$ the principal projection.*

Consider the Poisson bracket $\{\cdot,\cdot\}_{M/G_\nu}$ on M/G_ν described in Theorem 3.2. Then the map

$$\widetilde{\mathcal{O}} \to M/G_\nu, \qquad [x, Coad_g \nu] \mapsto [\phi_{g^{-1}} x]_\nu \tag{3.28}$$

is a Poisson diffeomorphism, where $\widetilde{\mathcal{O}}$ is equipped by the unique Poisson structure such that

$$\bar{\mu}_{\mathcal{O}} : (T^*\mathbb{R} \times_{\mathbb{R}} \widetilde{\mathcal{O}}, \Omega_{red}) \to \widetilde{\mathcal{O}}$$

is a Poisson epimorphism.

Proof. The Poisson manifold M/G_ν is isomorphic to the base space $(V^*\pi)_\nu = (J^{V^*\pi})^{-1}(\nu)/G_\nu$ of the reduced symplectic principal $\mathbb{R}$-bundle $(\mu_\pi)_\nu$ obtained by reducing μ_π at the level ν, where $(V^*\pi)_\nu$ is equipped with the unique Poisson structure which makes $(\mu_\pi)_\nu$ a Poisson epimorphism (see Theorem 3.2). The Poisson manifold $(V^*\pi)_\nu$ is isomorphic to the base space $(V^*\pi)_{\mathcal{O}} = (J^{V^*\pi})^{-1}(\mathcal{O})/G$ of the orbit reduced symplectic principal $\mathbb{R}$-bundle $(\mu_\pi)_{\mathcal{O}}$ (see Theorem 2.16). Finally, $(V^*\pi)_{\mathcal{O}}$ is isomorphic as Poisson manifold to the quotient manifold $\widetilde{\mathcal{O}} = (M \times \mathcal{O})/G$, where $\widetilde{\mathcal{O}}$ is equipped by the unique Poisson structure such that $\bar{\mu}_{\mathcal{O}}$ is a Poisson epimorphism.

Composing all the Poisson diffeomorphism, one may easily obtain just the map (3.28). □

Chapter 4

Non-autonomous Hamiltonian reduction

4.1 Non-autonomous Hamiltonian systems

In this chapter, we will extend the results in Section 1.3 for a symplectic principal $\mathbb{R}$-bundle in the presence of a Hamiltonian section. Let $\mu : A \to V$ be a principal $\mathbb{R}$-bundle with principal action $\psi : \mathbb{R} \times A \to A$ and infinitesimal generator Z_μ.

As in Section 2.1, if $h : V \to A$ is a section of the principal $\mathbb{R}$-bundle $\mu : A \to V$, a function $F_h : A \to \mathbb{R}$ is induced. Indeed, for any $a \in A$, since a and $h(\mu(a))$ are in the same $\mathbb{R}$-orbit, there exists a unique real number $F_h(a)$ such that

$$a = \psi\left(F_h(a), h(\mu(a))\right), \qquad \text{for any } a \in A, \tag{4.1}$$

(see [20]).

In the following result which will be useful in the sequel, we will prove two simple properties of the function $F_h : A \to \mathbb{R}$.

Lemma 4.1. *Let $\mu : A \to V$ be a principal $\mathbb{R}$-bundle with principal action $\psi : \mathbb{R} \times A \to A$ and $h : V \to A$ a section of μ. Then*

$$F_h(h(v)) = 0, \qquad \textit{for any } v \in V, \tag{4.2}$$

$$F_h(\psi(s, a)) = s + F_h(a), \qquad \textit{for any } s \in \mathbb{R},\ a \in A. \tag{4.3}$$

Proof. For any $v \in V$, we have (see (4.1))

$$h(v) = \psi\big(F_h(h(v)), h(\mu(h(v)))\big) = \psi(F_h(h(v)), h(v)).$$

Thus, since ψ is a free action, we deduce that $F_h(h(v)) = 0$.

Moreover, from (4.1), for any $s \in \mathbb{R}$ and $a \in A$

$$\psi(s,a) = \psi\big(F_h(\psi(s,a)), h(\mu(\psi(s,a)))\big) = \psi\big(F_h(\psi(s,a)), h(\mu(a))\big). \tag{4.4}$$

On the other hand, using again (4.1), it follows that

$$\psi(s,a) = \psi\big(s, \psi(F_h(a), h(\mu(a)))\big) = \psi\big(s + F_h(a), h(\mu(a))\big). \tag{4.5}$$

Comparing (4.4) and (4.5), we obtain (4.3). □

Note that, for a function $F \in C^\infty(A)$, the fact that $Z_\mu(F) = 1$ is equivalent to condition

$$F(\psi(s,a)) = s + F(a), \qquad \text{for any } s \in \mathbb{R} \text{ and } a \in A. \tag{4.6}$$

Indeed, if $Z_\mu(F) = 1$, then for any $a \in A$, the real functions $s \mapsto F(\psi(s,a))$ and $s \mapsto s$ have the same derivative. Thus, they differ for a constant. Since $\psi(0,a) = a$, condition (4.6) holds. The converse follows from the fact that the flow of Z_μ is $\{\psi_s\}$.

As a consequence, we obtain the following result which was stated firstly in [20].

Proposition 4.2. *There is a one-to-one correspondence between the space of sections $h : V \to A$ and the set*

$$\{F \in C^\infty(A) \,|\, Z_\mu(F) = 1\},$$

where $Z_\mu \in \mathfrak{X}(A)$ is the infinitesimal generator of the principal $\mathbb{R}$-bundle $\mu : A \to V$.

Proof. If $h : V \to A$ is a section of μ, then, from (4.3), it follows that the function F_h defined by (4.1) is such that $Z_\mu(F_h) = 1$. Conversely, if $F \in C^\infty(A)$ is such that $Z_\mu(F) = 1$, then we may define a map $h : V \to A$ as

$$h(v) = \psi(-F(a), a), \qquad \text{for any } v \in V,$$

where $a \in A$ is such that $\mu(a) = v$. Note that h is well-defined, because if $\mu(a) = \mu(a') = v$, with $a, a' \in A$, then, there exists $s \in \mathbb{R}$ such that $a' = \psi(s,a)$. It follows that

$$\psi(-F(a'), a') = \psi(-s - F(a), \psi(s,a)) = \psi(-F(a), a).$$

Moreover, h is clearly a section of μ. □

Furthermore, using (4.3), we deduce that

$$\psi_s^*(\mathrm{d}F_h) = \mathrm{d}F_h, \qquad \text{for any } s \in \mathbb{R}$$

and since $\mathrm{d}F_h(Z_\mu) = 1$ it follows that $\mathrm{d}F_h : TA \to \mathbb{R}$ is the connection 1-form of a principal connection on the principal $\mathbb{R}$-bundle $\mu : A \to V$ (see [20]).

The horizontal subbundle associated with the principal connection is

$$a \in A \mapsto H_a^h = \{X \in T_aA | X(F_h) = 0\} \subseteq T_aA$$

and thus

$$T_aA = H_a^h \oplus V_a\mu = H_a^h \oplus \langle Z_\mu(a)\rangle. \tag{4.7}$$

Note that

$$T_{h(\mu(a))}\mu\big((T_{\mu(a)}h \circ T_a\mu)(X)\big) = (T_a\mu)(X)$$

and, moreover, from (4.2)

$$\{(T_{\mu(a)}h \circ T_a\mu)(X)\}(F_h) = 0.$$

This implies that

$$(T_{h(\mu(a))}\psi_{F_h(a)})((T_{\mu(a)}h \circ T_a\mu)(X)) = X - X(F_h)Z_\mu(a)$$

and, therefore, the horizontal projector $\mathrm{hor}_a^h : T_aA \to H_a^h$ is given by

$$\mathrm{hor}_a^h(X) = (T_{h(\mu(a))}\psi_{F_h(a)} \circ T_{\mu(a)}h \circ T_a\mu)(X). \tag{4.8}$$

Definition 4.3. *A* non-autonomous Hamiltonian system (A, μ, Ω, h) *is a symplectic principal* $\mathbb{R}$*-bundle* $\mu : (A, \Omega) \to V$ *endowed with a section* $h : V \to A$ *of* μ*, i.e. a smooth map such that* $\mu \circ h = id_V$*.*

The section $h : V \to A$ *is called the* Hamiltonian section of the system.

In this section we will prove that, given a non-autonomous Hamiltonian system (A, μ, Ω, h), the base manifold V of the principal $\mathbb{R}$-bundle μ may be equipped with a cosymplectic structure.

Proposition 4.4. *Let* (A, μ, Ω, h) *be a non-autonomous Hamiltonian system. Then the Hamiltonian vector field* $\mathcal{H}_{F_h} \in \mathfrak{X}(A)$ *of the function* F_h *associated with a Hamiltonian section* $h : V \to A$ *is* μ*-projectable to a vector field* $\mathcal{R}_h$ *on* V*.*

Proof. Since the infinitesimal generator Z_μ is locally Hamiltonian, for any $a \in A$, there exists a function τ defined on an open neighbourhood U of

a such that the restriction of Z_μ to U is the Hamiltonian vector field of τ. Using the definition of F_h, we have that on U

$$\{\tau, F_h\}_A = -\mathcal{H}_\tau(F_h) = -Z_\mu(F_h) = -1,$$

$\{\cdot,\cdot\}_A$ being the Poisson bracket on A induced by the symplectic form Ω. As a consequence,

$$\mathcal{L}_{Z_\mu}\mathcal{H}_{F_h} = [\mathcal{H}_\tau, \mathcal{H}_{F_h}] = -\mathcal{H}_{\{\tau,F_h\}_A} = 0.$$

Thus, the Lie derivative of $\mathcal{H}_{F_h}$ with respect to any vertical vector field is again vertical. This is a sufficient (and necessary) condition to ensure the μ-projectability of $\mathcal{H}_{F_h}$. □

The μ-projection $\mathcal{R}_h$ of $\mathcal{H}_{F_h}$ is a vector field on V, which describes the Hamiltonian dynamics of the non-autonomous Hamiltonian system (A, μ, Ω, h), as we will see in what follows.

Proposition 4.5. *Let (A, μ, Ω, h) be a non-autonomous Hamiltonian system. If $\omega_h \in \Omega^2(V)$ and $\eta_h \in \Omega^1(V)$ are defined by*

$$\omega_h = h^*\Omega, \qquad \eta_h = -h^*(i_{Z_\mu}\Omega), \tag{4.9}$$

then

$$\Omega = \mu^*\omega_h - \mathrm{d}F_h \wedge \mu^*\eta_h \tag{4.10}$$

and

$$\mu^*\eta_h = -i_{Z_\mu}\Omega. \tag{4.11}$$

Proof. First of all, we will see that (4.11) holds. Using that Z_μ is μ-vertical, it is clear that

$$(\mu^*\eta_h)(Z_\mu) = -(i_{Z_\mu}\Omega)(Z_\mu) = 0. \tag{4.12}$$

On the other hand, from (4.8) and since $\mu \circ \psi_s = \mu$ and $\mu \circ h = id$, it follows that

$$(\mu^*\eta_h)(a)(\mathrm{hor}^h_a(X)) = -[h^*(i_{Z_\mu}\Omega)](\mu(a))((T_a\mu)(X)),$$

for $a \in A$ and $X \in T_aA$.

Thus, since ψ is a symplectic action, we obtain that

$$\mu^*(\eta_h)(a)(\mathrm{hor}^h_a(X)) = -(i_{Z_\mu}\Omega)(a)(\mathrm{hor}^h_a(X)).$$

This, using (4.7) and (4.12), proves (4.11).

Next, we will see that (4.10) holds. From (4.11) and since Z_μ is vertical, we deduce that

$$i_{Z_\mu}\Omega = i_{Z_\mu}(\mu^*\omega_h - \mathrm{d}F_h \wedge \mu^*\eta_h). \tag{4.13}$$

On the other hand, using (4.8) and (4.9) and the fact that ψ is a symplectic action, we have that

$$\begin{aligned}\Omega(a)(\mathrm{hor}_a^h(X),\mathrm{hor}_a^h(Y)) &= (\mu^*\omega_h)(a)(X,Y)\\ &= (\mu^*\omega_h)(a)(\mathrm{hor}_a^h(X),\mathrm{hor}_a^h(Y))\\ &= (\mu^*\omega_h - \mathrm{d}F_h\wedge\mu^*\eta_h)(\mathrm{hor}_a^h(X),\mathrm{hor}_a^h(Y))\end{aligned}\tag{4.14}$$

for $a \in A$ and $X, Y \in T_aA$.

Therefore, from (4.7), (4.13) and (4.14), we deduce (4.10). $\square$

Now, we may prove the following result.

Theorem 4.6. *Let (A,μ,Ω,h) be a non-autonomous Hamiltonian system with infinitesimal generator Z_μ. If ω_h and η_h are the 1-form and the 2-form respectively, on V defined by (4.9), then (V,ω_h,η_h) is a cosymplectic manifold. The Reeb vector field of the cosymplectic structure on V is just $\mathcal{R}_h$.*

Proof. From (4.9) and (4.11) and since Ω is closed and $\mathcal{L}_{Z_\mu}\Omega = 0$, we deduce that ω_h and η_h are closed.

Now, using (4.11) and Proposition 4.4, we have that

$$\eta_h(\mathcal{R}_h) = (\mu^*\eta_h)(\mathcal{H}_{F_h}) = (i_{\mathcal{H}_{F_h}}\Omega)(Z_\mu) = 1. \tag{4.15}$$

On the other hand, from (4.10) and Proposition 4.4, it follows that

$$\mu^*(i_{\mathcal{R}_h}\omega_h) = i_{\mathcal{H}_{F_h}}(\Omega + \mathrm{d}F_h\wedge\mu^*\eta_h).$$

Thus, using (4.15), we obtain that $\mu^*(i_{\mathcal{R}_h}\omega_h) = 0$ which implies that

$$i_{\mathcal{R}_h}\omega_h = 0. \tag{4.16}$$

Next, suppose that $\dim A = 2n+2$. Then, from (4.16), we deduce that $\mathrm{rank}(\omega_h) \leq 2n$. Therefore, using (4.10), it follows that

$$0 \neq \Omega^{n+1} = c(\mu^*\omega_h)^n \wedge \mathrm{d}F_h \wedge \mu^*\eta_h, \text{ with } c\in\mathbb{R},\ \ c\neq 0.$$

Consequently, the rank of $\mu^*\omega_h$ is $2n$ and we have that

$$\mathrm{rank}\ \omega_h = 2n. \tag{4.17}$$

Conditions (4.15), (4.16) and (4.17) imply that $\eta_h \wedge \omega_h^n \neq 0$. $\square$

The cosymplectic structure (ω_h, η_h) on V defined on the base manifold V of a non-autonomous Hamiltonian system (A, μ, Ω, h) induces a Poisson structure $\{\cdot,\cdot\}_h$ on V. On the other hand, as we know (see Proposition 2.6), V is equipped with a Poisson structure $\{\cdot,\cdot\}_V$ in such a way that μ is a Poisson map. The next result shows that the Poisson brackets $\{\cdot,\cdot\}_h$ and $\{\cdot,\cdot\}_V$ are equal.

Proposition 4.7. *Let (A, μ, Ω, h) be a non-autonomous Hamiltonian system, $\{\cdot,\cdot\}_h$ the Poisson bracket on V associated with the cosymplectic structure (ω_h, η_h) and $\{\cdot,\cdot\}_V$ the Poisson bracket on V induced by the symplectic principal $\mathbb{R}$-bundle structure. Then, $\{\cdot,\cdot\}_h = \{\cdot,\cdot\}_V$.*

Proof. Fix a real C^∞-function f on V. It is sufficient to prove that the Hamiltonian vector field X_f on V with respect to the Poisson bracket $\{\cdot,\cdot\}_V$ is equal to the Hamiltonian vector field of f with respect to the cosymplectic structure (ω_h, η_h). Note that, since μ is a Poisson map, it follows that the Hamiltonian vector field $\mathcal{H}_{f\circ\mu} \in \mathfrak{X}(A)$ is μ-projectable and its projection is just X_f. Thus, from (4.11), we have

$$\eta_h(X_f) = \mu^*\eta_h(\mathcal{H}_{f\circ\mu}) = -i_{Z_\mu}\Omega(\mathcal{H}_{f\circ\mu}) = Z_\mu(f\circ\mu) = 0. \tag{4.18}$$

On the other hand, using that $\mathcal{H}_{F_h}$ is μ-projectable on $\mathcal{R}_h$, we deduce that

$$\mathcal{R}_h(f) = \mathcal{H}_{F_h}(f\circ\mu) = -\mathrm{d}F_h(\mathcal{H}_{f\circ\mu}).$$

Now, from (4.10) and (4.18), it follows that

$$\begin{aligned}(i_{X_f}\omega_h)(\mu(a))(T_a\mu(\bar{Y})) &= (\mu^*\omega_h)(a)(\mathcal{H}_{f\circ\mu}(a), \bar{Y})\\ &= \Omega(a)(\mathcal{H}_{f\circ\mu}(a), \bar{Y}) + (\mathrm{d}F_h \wedge \mu^*\eta_h)(a)(\mathcal{H}_{f\circ\mu}(a), \bar{Y})\\ &= (\mathrm{d}(f\circ\mu))(a)(\bar{Y}) + (\mathrm{d}F_h)(a)(\mathcal{H}_{f\circ\mu}(a))(\mu^*\eta_h)(a)(\bar{Y})\\ &= (\mathrm{d}f - \mathcal{R}_h(f)\eta_h)(\mu(a))(T_a\mu(\bar{Y})),\end{aligned}$$

for all $\bar{Y} \in T_aA$, with $a \in A$. Therefore,

$$i_{X_f}\omega_h = df - \mathcal{R}_h(f)\eta_h.$$

This ends the proof of the result. □

In what follows, we will prove that the integral curves of the vector field $\mathcal{R}_h$ satisfy local equations which are just the Hamilton equations. For this purpose, we will use canonical coordinates on the symplectic principal $\mathbb{R}$-bundle $\mu : (A, \Omega) \to M$ (see Theorem 2.7).

Let (t, p, q^i, p_i) be canonical coordinates on A. Suppose that the local expression of the Hamiltonian section $h : V \to A$ is

$$h(t, q^i, p_i) = (t, -H(t, q^j, p_j), q^i, p_i),$$

where H is a local function on V. Then, $F_h : A \to \mathbb{R}$ may be described locally by

$$F_h(t, p, q^i, p_i) = p + H(t, q^i, p_i).$$

Since (t, p, q^i, p_i) are Darboux coordinates for the symplectic form Ω on A, we have that

$$\begin{aligned}
\mathcal{H}_{F_h} &= \frac{\partial}{\partial t} - \frac{\partial H}{\partial t}\frac{\partial}{\partial p} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i},\\
\mathcal{R}_h &= \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}.
\end{aligned}$$

Finally, the cosymplectic structure (ω_h, η_h) on V is locally described by

$$\begin{aligned}
\omega_h &= -\frac{\partial H}{\partial q^i}\mathrm{d}t \wedge \mathrm{d}q^i - \frac{\partial H}{\partial p_i}\mathrm{d}t \wedge \mathrm{d}p_i + \mathrm{d}q^i \wedge \mathrm{d}p_i,\\
\eta_h &= \mathrm{d}t.
\end{aligned}$$

Thus, a curve on V with local expression

$$t \mapsto (t, q^i(t), p_i(t))$$

is an integral curve of $\mathcal{R}_h$ if and only if it satisfies the Hamilton equations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}.$$

Therefore, in the particular case when μ is the standard principal $\mathbb{R}$-bundle associated with a fibration $\pi : M \to \mathbb{R}$, we recover the results in Section 2.1.

4.2 Non-autonomous Hamiltonian systems and morphisms

In this section, we will discuss the behavior of non-autonomous Hamiltonian systems in the precence of symplectic principal $\mathbb{R}$-bundle morphisms.

We will suppose that two non-autonomous Hamiltonian systems (A, μ, Ω, h) and (A', μ', Ω', h') are given and that $\varphi : A \to A'$ is a symplectic principal $\mathbb{R}$-bundle morphism (see Section 2.2).

Definition 4.8. *The symplectic principal $\mathbb{R}$-bundle morphism $\varphi\colon A \to A'$ is said to be a morphism of non-autonomous Hamiltonian systems if h and h' are φ-related, that is*

$$\varphi \circ h = h' \circ \varphi^V, \tag{4.19}$$

where $\varphi^V : V \to V'$ is the induced map between the base manifolds.

An equivalent definition involves the corresponding functions $F_h : A \to \mathbb{R}$ and $F_{h'} : A' \to \mathbb{R}$.

Proposition 4.9. *The Hamiltonian sections $h : V \to A$ and $h' : V' \to A'$ are φ-related if and only if $F_h = F_{h'} \circ \varphi$.*

Proof. Using (2.7) and (4.1), we have that for any $a \in A$

$$\varphi(a) = \psi'(F_{h'}(\varphi(a)), h'(\mu'(\varphi(a)))) = \psi'(F_{h'}(\varphi(a)), h'(\varphi^V(\mu(a)))). \tag{4.20}$$

On the other hand, applying φ to both of the sides of (4.1) and using (2.5), we get

$$\varphi(a) = \varphi(\psi(F_h(a), h(\mu(a)))) = \psi'(F_h(a), \varphi(h(\mu(a)))). \tag{4.21}$$

Comparing (4.20) and (4.21), one may obtain the required equivalence. □

As a consequence, we have the following result.

Corollary 4.10. *Let $\mu : (A, \Omega) \to V$ and $\mu' : (A', \Omega') \to V'$ be symplectic principal $\mathbb{R}$-bundles and $h' : V' \to A'$ be a Hamiltonian section of μ'. If $\varphi : A \to A'$ is a symplectic principal $\mathbb{R}$-bundle morphism, then there exists a unique Hamiltonian section $h : V \to A$ such that φ is a morphism between the non-autonomous Hamiltonian systems (A, μ, Ω, h) and (A', μ', Ω', h').*

Proof. If we denote by $F = F_{h'} \circ \varphi : A \to \mathbb{R}$, since Z_μ and $Z_{\mu'}$ are φ-related, we have that for any $a \in A$

$$Z_\mu(a)(F) = T_a\varphi(Z_\mu(a))(F_{h'}) = Z_{\mu'}(\varphi(a))(F_{h'}) = 1.$$

Thus, F defines a Hamiltonian section $h : V \to A$ such that $F = F_h$. In view of Proposition 4.9, h and h' are φ-related. The unicity follows directly from Propositions 4.2 and 4.9. □

Finally, we will prove that, in the precence of a morphism of non-autonomous Hamiltonian systems, one may produce solutions of the dynamics for the second system from solutions of the dynamics for the first one.

Theorem 4.11. *Let $\varphi : A \to A'$ be a symplectic principal $\mathbb{R}$-bundle morphism between the non-autonomous Hamiltonian systems (A, μ, Ω, h) and (A', μ', Ω', h'). Denote by $\varphi^V : V \to V'$ the corresponding map between the base spaces V and V' and consider the induced cosymplectic structures (ω_h, η_h) and $(\omega_{h'}, \eta_{h'})$ on V and V', respectively.*

If h and h' are φ-related, then φ^V is a cosymplectic map, i.e.

$$(\varphi^V)^*\omega_{h'} = \omega_h, \qquad (\varphi^V)^*\eta_{h'} = \eta_h.$$

In particular, $\mathcal{R}_h \in \mathfrak{X}(V)$ and $\mathcal{R}_{h'} \in \mathfrak{X}(V')$ are φ^V-related, that is

$$T_v\varphi^V(\mathcal{R}_h(v)) = \mathcal{R}_{h'}(\varphi^V(v)), \qquad \text{for any } v \in V. \tag{4.22}$$

Thus, if $\gamma\colon I \to V$ is a solution of the Hamilton equations for the non-autonomous Hamiltonian system (A, μ, Ω, h) then $\varphi^V \circ \gamma\colon I \to V'$ is a solution of the Hamilton equations for the non-autonomous Hamiltonian system (A', μ', Ω', h').

Proof. Since φ is a symplectic morphism, using (4.9) and (4.19), we have

$$(\varphi^V)^*\omega_{h'} = (\varphi^V)^*(h'^*\Omega') = h^*(\varphi^*\Omega') = h^*\Omega = \omega_h.$$

Moreover, from (4.9) and Proposition (2.9), we get

$$(\varphi^V)^*\eta_{h'} = -(\varphi^V)^*(h'^*(i_{Z_{\mu'}}\Omega')) = -h^*(\varphi^*(i_{Z_{\mu'}}\Omega')) = -h^*(i_{Z_\mu}\Omega) = \eta_h.$$

Thus, φ^V is a cosymplectic morphism. □

Another way to prove (4.22) is to use the fact that $\mathcal{R}_h$ (respectively, $\mathcal{R}_{h'}$) is μ-projectable (respectively, μ'-projectable) of the Hamiltonian vector field of F_h (respectively, $F_{h'}$). Indeed, for any $v \in V$, with $v = \mu(a)$, since φ preserves the symplectic forms and using (2.7), we get

$$\begin{aligned} T_v\varphi^V(\mathcal{R}_h(v)) &= T_v\varphi^V(T_a\mu(\mathcal{H}_{F_h}(a))) = T_{\varphi(a)}\mu'(T_a\varphi(\mathcal{H}_{F_h}(a))) \\ &= T_{\varphi(a)}\mu'(\mathcal{H}_{F_{h'}}(\varphi(a))) = \mathcal{R}_{h'}(\varphi^V(v)). \end{aligned}$$

4.3 Non-autonomous Hamiltonian reduction Theorem

In this section, we will obtain a reduction theorem for non-autonomous Hamiltonian systems.

Let $\mu : (A, \Omega) \to V$ be a symplectic principal $\mathbb{R}$-bundle with infinitesimal generator Z_μ and $\phi : G \times A \to A$ a canonical action of a Lie group G on the symplectic principal $\mathbb{R}$-bundle $\mu : (A, \Omega) \to V$. Denote by $\phi^V : G \times V \to V$ the corresponding action on V.

Now suppose that $h : V \to A$ is a Hamiltonian section of μ.

Definition 4.12. *The Hamiltonian section h is said to be G-*equivariant *if h is equivariant with respect to the actions ϕ and ϕ^V, that is,*

$$h \circ \phi_g^V = \phi_g \circ h, \qquad \textit{for } g \in G.$$

In other words, h is G-equivariant if h is ϕ_g is a morphism of non-autonomous Hamiltonian systems, for all $g \in G$.

Equivalently, in view of Proposition 4.9, h is G-equivariant if the corresponding function F_h is G-invariant, i.e. $F_h \circ \phi_g = F_h$ for any $g \in G$.

Proposition 4.13. *If $h : V \to A$ is a G-equivariant Hamiltonian section, the induced action $\phi^V : G \times V \to V$ is a cosymplectic action with respect to the cosymplectic structure (ω_h, η_h) on V defined by h.*

Moreover, if $J : A \to \mathfrak{g}^*$ is a momentum map with respect to the action ϕ, then the induced Poisson momentum map $J^V : V \to \mathfrak{g}^*$ is such that $\mathcal{R}_h(J_\xi^V) = 0$, for any $\xi \in \mathfrak{g}$.

Proof. Since ϕ_g is a morphism of non-autonomous Hamiltonian systems for any $g \in G$, the first part follows easily from Theorem 4.11. Moreover, for any $\xi \in \mathfrak{g}$ we have (see (2.18))

$$\begin{aligned} \mathcal{R}_h(J_\xi^V) &= \mathcal{H}_{F_h}(J_\xi^V \circ \mu) = \mathcal{H}_{F_h}(J_\xi) \\ &= -\mathcal{H}_{J_\xi}(F_h) = -\xi_A(F_h) = 0, \end{aligned}$$

ξ_A being the infinitesimal generator of ϕ defined by ξ. The last equality follows from the G-invariance of F_h. □

As a consequence, we directly deduce the following result.

Corollary 4.14. *Let (A, μ, Ω, h) be a non-autonomous Hamiltonian system and $\phi \colon G \times A \to A$ a canonical action on the symplectic principal $\mathbb{R}$-bundle $\mu \colon (A, \Omega) \to V$. If $h : V \to A$ is G-equivariant, then the Reeb vector field $\mathcal{R}_h \in \mathfrak{X}(V)$ of the corresponding cosymplectic structure (ω_h, η_h) on V is ϕ^V-invariant, where $\phi^V : G \times V \to V$ is the induced action.*

Now, we are able to reduce the non-autonomous Hamiltonian system.

Let (A, μ, Ω, h) be a non-autonomous Hamiltonian system equipped with a canonical action $\phi : G \times A \to A$ of a Lie group G on the manifold A with an Ad^*-equivariant momentum map $J : A \to \mathfrak{g}^*$. Suppose that the induced action $\phi^V : G \times V \to V$ on V is free and proper. Let ν be an element of $\mathfrak{g}^*$. Then we induce a free and proper action $\phi^V : G \times (J^V)^{-1}(\nu) \to (J^V)^{-1}(\nu)$ of G on $(J^V)^{-1}(\nu)$ and, using Proposition 4.13, we may apply Albert's Theorem (see Theorem 1.10). Thus, we reduce the cosymplectic manifold (V, ω_h, η_h)

for obtaining a reduced cosymplectic manifold $(V_\nu, (\omega_h)_\nu, (\eta_h)_\nu)$, where V_ν is the quotient manifold $(J^V)^{-1}(\nu)/G_\nu$ and $(\omega_h)_\nu$, $(\eta_h)_\nu$ are the 2-form and 1-form on V_ν characterized by

$$(\pi_\nu^V)^*(\omega_h)_\nu = (i_\nu^V)^*\omega_h, \qquad (\pi_\nu^V)^*(\eta_h)_\nu = (i_\nu^V)^*\eta_h \tag{4.23}$$

$\pi_\nu^V : (J^V)^{-1}(\nu) \to V_\nu$ and $i_\nu^V : (J^V)^{-1}(\nu) \hookrightarrow V$ being the canonical projection and the canonical inclusion, respectively.

On the other hand, from Theorem 2.13, we obtain a reduced symplectic principal $\mathbb{R}$-bundle $\mu_\nu : (A_\nu, \Omega_\nu) \to V_\nu$ with infinitesimal generator Z_{μ_ν}. We recall that the restriction of Z_μ to $J^{-1}(\nu)$ is tangent to $J^{-1}(\nu)$ and that $Z_{\mu|J^{-1}(\nu)}$ is π_ν-projectable on Z_{μ_ν}. Moreover, the momentum map $J : A \to \mathfrak{g}^*$ and the G-invariant Hamiltonian section h satisfy the following property (see (2.18))

$$h((J^V)^{-1}(\nu)) \subset J^{-1}(\nu). \tag{4.24}$$

Thus, we may define the function $h : (J^V)^{-1}(\nu) \to J^{-1}(\nu)$, which induces a smooth map $h_\nu : V_\nu \to A_\nu$ given by

$$h_\nu(\pi_\nu^V(v)) = \pi_\nu(h(v)), \qquad \text{with } v \in (J^V)^{-1}(\nu). \tag{4.25}$$

The function h_ν is a section of μ_ν, so h_ν is a Hamiltonian section of the symplectic principal $\mathbb{R}$-bundle $\mu_\nu : A_\nu \to V_\nu$ and $(A_\nu, \mu_\nu, \Omega_\nu, h_\nu)$ is a non-autonomous Hamiltonian system.

A direct computation, using (2.20), (2.21), (4.1) and (4.25), shows that the function F_{h_ν} on $A_\nu = J^{-1}(\nu)/G_\nu$ is characterized by the following condition

$$F_{h_\nu} \circ \pi_\nu = F_{h|J^{-1}(\nu)}. \tag{4.26}$$

Thus, F_{h_ν} may be obtained from F_h by passing to quotient.

Moreover, from Theorem 4.6, $(V_\nu, (\omega_\nu)_{h_\nu}, (\eta_\nu)_{h_\nu})$ is a cosymplectic manifold whose structure is given by

$$(\omega_\nu)_{h_\nu} = h_\nu^*\Omega_\nu, \qquad (\eta_\nu)_{h_\nu} = -h_\nu^*(i_{Z_{\mu_\nu}}\Omega_\nu). \tag{4.27}$$

Theorem 4.15. *Let (A, μ, Ω, h) be a non-autonomous Hamiltonian system and $\phi : G \times A \to A$ be a canonical action of G on A such that the induced action on V is free and proper. Suppose that $J : A \to \mathfrak{g}^*$ is an Ad^*-equivariant momentum map. If h is G-equivariant, then, for any $\nu \in \mathfrak{g}^*$, the cosymplectic structure $((\omega_\nu)_{h_\nu}, (\eta_\nu)_{h_\nu})$ on V_ν, induced by the reduced non-autonomous Hamiltonian system $(A_\nu, \mu_\nu, \Omega_\nu, h_\nu)$, is the one deduced from Albert's reduction of the cosymplectic structure (ω_h, η_h) on V. In other words,*

$$(\omega_\nu)_{h_\nu} = (\omega_h)_\nu, \qquad (\eta_\nu)_{h_\nu} = (\eta_h)_\nu. \tag{4.28}$$

In particular, the dynamics $\mathcal{R}_{h_\nu}$ of the reduced non-autonomous Hamiltonian system is just the π_ν^V-projection of the restriction to $(J^V)^{-1}(\nu)$ of the dynamics $\mathcal{R}_h$ of (A,μ,Ω,h). More precisely,

$$\mathcal{R}_{h_\nu}(\pi_\nu^V(v)) = T_v\pi_\nu^V(\mathcal{R}_h(v)), \qquad \textit{for any } v\in (J^V)^{-1}(\nu). \tag{4.29}$$

Thus, if $\gamma\colon I\to V$ is a solution of the Hamilton equations for the non-autonomous Hamiltonian system (A,μ,Ω,h), $0\in I$ and $\gamma(0)\in (J^V)^{-1}(\nu)$, then $\sigma(I)\subseteq (J^V)^{-1}(\nu)$ and $\pi_\nu^V\circ\gamma\colon I\to V_\nu$ is a solution of the Hamilton equations for the reduced non-autonomous Hamiltonian system $(A_\nu,\mu_\nu,\Omega_\nu,h_\nu)$.

Proof. In order to show (4.28), we will prove that the 2-form $(\omega_\nu)_{h_\nu}$ and the 1-form $(\eta_\nu)_{h_\nu}$ satisfy condition (4.23). In fact, if $v\in (J^V)^{-1}(\nu)$ and $X,Y\in T_v((J^V)^{-1}(\nu))$, then, using (1.21) and (4.25), we have that

$$\begin{aligned}
\big((\pi_\nu^V)^*(\omega_\nu)_{h_\nu}\big)(v)(X,Y) &= (\omega_\nu)_{h_\nu}(\pi_\nu^V(v))(T_v\pi_\nu^V(X),T_v\pi_\nu^V(Y))\\
&= \Omega_\nu(h_\nu(\pi_\nu^V(v)))\left(T_{\pi_\nu^V(v)}h_\nu(T_v\pi_\nu^V(X)),T_{\pi_\nu^V(v)}h_\nu(T_v\pi_\nu^V(Y))\right)\\
&= \Omega_\nu(\pi_\nu(h(v)))\left(T_{h(v)}\pi_\nu(T_vh(X)),T_{h(v)}\pi_\nu(T_vh(Y))\right)\\
&= (\pi_\nu^*\Omega_\nu)(h(v))\,(T_vh(X),T_vh(Y))\\
&= (i_\nu^*\Omega)(h(v))\,(T_vh(X),T_vh(Y)) = ((i_\nu^V)^*\omega_h)(v)(X,Y).
\end{aligned}$$

Thus, $(\pi_\nu^V)^*(\omega_\nu)_{h_\nu} = (i_\nu^V)^*\omega_h$. Moreover, using again (1.21) and (4.25), we deduce that

$$\begin{aligned}
\big((\pi_\nu^V)^*(\eta_\nu)_{h_\nu}\big)(v)(X) &= -\left(h_\nu^*(i_{Z_{\mu_\nu}}\omega_\nu)\right)(\pi_\nu^V(v))\left(T_v\pi_\nu^V(X)\right)\\
&= -\left(i_{Z_{\mu_\nu}}\Omega_\nu\right)(h_\nu(\pi_\nu^V(v)))\left(T_{\pi_\nu^V(v)}h_\nu(T_v\pi_\nu^V(X)\right)\\
&= -\left(i_{Z_{\mu_\nu}}\Omega_\nu\right)(\pi_\nu(h(v)))\left(T_{h(v)}\pi_\nu(T_vh(X))\right)\\
&= -\left(i_{Z_{\mu|J^{-1}(\nu)}}(i_\nu^*\Omega)\right)(h(v))(T_vh(X))\\
&= -\left(i_\nu^*(i_{Z_\mu}\Omega)\right)(h(v))(T_vh(X)) = ((i_\nu^V)^*\eta_h)(v)(X).
\end{aligned}$$

Therefore, $(\pi_\nu^V)^*(\eta_\nu)_{h_\nu} = (i_\nu^V)^*\eta_h$. □

Note that another way to obtain (4.29) is to use (4.26). In fact, since $\mathcal{H}_{F_h}$ (respectively, $\mathcal{H}_{F_{h_\mu}}$) is μ-projectable (respectively, μ_ν-projectable) on $\mathcal{R}_h$ (respectively, $\mathcal{R}_{h_\nu}$), using (1.26), we have that

$$\begin{aligned}
\mathcal{R}_{h_\nu}(\pi_\nu^V(v)) &= T_{\pi_\nu(a)}\mu_\nu(\mathcal{H}_{F_{h_\mu}}(\pi_\nu(a))) = T_{\pi_\nu(a)}\mu_\nu\left(T_a\pi_\nu(\mathcal{H}_{F_h}(a))\right)\\
&= T_{\mu(a)}\pi_\nu^V\left(T_a\mu(\mathcal{H}_{F_h}(a))\right) = T_{\mu(a)}\pi_\nu^V(\mathcal{R}_h(v)),
\end{aligned}$$

for any $v=\mu(a)\in (J^V)^{-1}(\nu)$, with $a\in J^{-1}(\nu)$.

4.4 Recontruction of the unreduced dynamics from the reduced dynamics

In this last part of the Chapter, we will develop a reconstruction procedure for the dynamics of a given non-autonomous Hamiltonian system from that of the reduced system. We will present a similar procedure to that given in [41].

We will use the following general set-up. Let (A, μ, Ω, h) be a non-autonomous Hamiltonian system and $\phi : G \times A \to A$ be a canonical action of a connected Lie group G on the symplectic principal $\mathbb{R}$-bundle $\mu : (A, \Omega) \to V$ with Ad^*-equivariant momentum map $J : A \to \mathfrak{g}^*$. Under regularity hypotheses, if h is G-equivariant and $\nu \in \mathfrak{g}^*$, we know that a reduced non-autonomous Hamiltonian system $(A_\nu, \mu_\nu, \Omega_\nu, h_\nu)$ is given (see Theorem 4.15).

Recall that, if (A, μ, Ω, h) is a non-autonomous Hamiltonian system, then its dynamics is represented by the Reeb vector field $\mathcal{R}_h$ of the cosymplectic structure (ω_h, η_h) on the base space V. Indeed, its integral curves are just the solutions of the Hamilton equations (see Section 4.1).

Moreover, if $\gamma\colon I \to V$ is an integral curve of $\mathcal{R}_h$ with initial condition in $(J^V)^{-1}(\nu)$ then $\gamma(I) \subseteq (J^V)^{-1}(\nu)$ and $\pi^V_\nu \circ \gamma\colon I \to V_\nu$ is a solution of the reduced non-auotnomous Hamiltonian system, where $\pi^V_\nu\colon (J^V)^{-1}(\nu) \to V_\nu$ is the projection. In other words, $\pi^V_\nu \circ \gamma$ is an integral curve of the reduced Reeb vector field $\mathcal{R}_{h_\nu}$.

Next, we will discuss the inverse procedure. We will start from an integral curve $c_\nu(t)$ of the Reeb vector field $\mathcal{R}_{h_\nu}$ such that $c_\nu(0) = \pi^V_\nu(v_0)$, with $v_o \in (J^V)^{-1}(\nu)$. Our goal is to construct an integral curve $c(t)$ of the Reeb vector field of the unreduced non-autonomous Hamiltonian system (A, μ, Ω, h). Moreover, we require that $c(t)$ projects on $c_\nu(t)$ and that c passes thought v_0, i.e.

$$c(t) \in (J^V)^{-1}(\nu), \quad \pi^V_\nu(c(t)) = c_\nu(t), \quad c(0) = v_0.$$

Pick an arbitrary smooth curve $d(t)$ whose support is contained in $(J^V)^{-1}(\nu)$ such that $d(0) = v_0$ and $\pi^V_\nu(d(t)) = c_\nu(t)$. We write

$$c(t) = \phi^V_{g(t)}(d(t)) \tag{4.30}$$

for some curve $g(t)$ on G_ν to be determined. We have the following result

Theorem 4.16. *Under the previous assumptions, the curve $c(t)$ is an integral curve of $\mathcal{R}_h$ if and only if*

$$\dot{d}(t) + (T_{g(t)} l_{g(t)^{-1}}(\dot{g}(t)))_V(d(t)) = \mathcal{R}_h(d(t)), \tag{4.31}$$

where $l_g : G \to G$ denotes the left translation in the Lie group G, with $g \in G$, and $(T_{g(t)} l_{g(t)^{-1}}(\dot g(t)))_V$ is the infinitesimal generator of $T_{g(t)} l_{g(t)^{-1}}(\dot g(t)) \in \mathfrak{g}$ with respect to the action ϕ^V.

Proof. From (4.30), we deduce that

$$\begin{aligned}\dot c(t) &= T_{d(t)}\phi^V_{g(t)}(\dot d(t)) + T_{g(t)}\phi^V_{d(t)}(\dot g(t))\\ &= T_{d(t)}\phi^V_{g(t)}(\dot d(t)) + T_{d(t)}\phi^V_{g(t)}\left((T_{g(t)} l_{g(t)^{-1}}(\dot g(t)))_V(d(t))\right).\end{aligned}$$

Thus, applying $T_{c(t)}\phi^V_{g(t)^{-1}}$ to both sides, we obtain

$$T_{c(t)}\phi^V_{g(t)^{-1}}(\dot c(t)) = \dot d(t) + (T_{g(t)} l_{g(t)^{-1}}(\dot g(t)))_V(d(t)). \tag{4.32}$$

On the other hand, using Corollary 4.14, $\mathcal{R}_h$ is ϕ^V-invariant, that is

$$T_v\phi^V_g(\mathcal{R}_h(v)) = \mathcal{R}_h(\phi^V_g(v)), \quad \text{for } g \in G \text{ and } v \in V. \tag{4.33}$$

Comparing (4.32) and (4.33), we obtain the required equivalence. □

Since $d(t)$ is *a priori* fixed, (4.31) is an equation written in terms of $g(t)$ only. One may solve it in two steps:

1) we find a curve $t \mapsto \xi(t)$ is $\mathfrak{g}$ such that

$$(\xi(t))_V(d(t)) = \mathcal{R}_h(d(t)) - \dot d(t); \tag{4.34}$$

2) with $\xi(t)$ determined, we solve the ordinary differential equation in G_ν

$$\dot g(t) = T_e L_{g(t)}(\xi(t)), \tag{4.35}$$

with initial condition

$$g(0) = e.$$

It's clear that, if $\xi(t)$ and $g(t)$ are solutions of (4.34) and (4.35), respectively, then the curve $c(t)$ given by (4.30) is an integral curve of $\mathcal{R}_h$ with the required conditions.

Note that this procedure *depends* on the choice of the curve $d(t)$ on $(J^V)^{-1}(\nu)$.

Chapter 5

Examples

5.1 Reduction of some non-autonomous Hamiltonian systems defined by reductive Lie groups

In Section 3.2, we have studied the case in which the initial surjective submersion $\pi\colon M \to \mathbb{R}$ is the projection of a principal G-bundle over the real line. In such a case, we have described a canonical Poisson structure on the space M/G_ν, where G_ν is the isotropy group of an element $\nu \in \mathfrak{g}^*$. This description has been obtained using a Poisson diffeomorphism between M/G_ν and the base space of a suitable symplectic principal $\mathbb{R}$-bundle. As we remarked, in such a case the fibration π is trivializable, although a canonical trivialization is not given.

In this section, we will suppose that a trivialization of the initial fibration π is fixed, that is $\pi\colon G \times \mathbb{R} \to \mathbb{R}$ is just the canonical projection on the second factor. This hypothesis is quite usual in Geometric Mechanics, as we will show in Sections 5.2 and 5.3.

In what follows, we will use the following set-up. Let G be a (connected) Lie group and K be a closed Lie subgroup of G. Consider the following (left) action

$$\phi\colon K \times (G \times \mathbb{R}) \to G \times \mathbb{R}, \qquad (k,(g,t)) \mapsto (kg,t).$$

Clearly, ϕ is a free and proper action of K on $G \times \mathbb{R}$. As a consequence, K acts by cotangent lifts on the cotangent bundle $T^*(G \times \mathbb{R}) \simeq T^*G \times \mathbb{R}^2$. In body coordinates (see Section 1.4), the cotangent lift action is given by

$$T^*\phi\colon K \times (G \times \mathfrak{g}^* \times \mathbb{R}^2) \to G \times \mathfrak{g}^* \times \mathbb{R}^2, \qquad (k,(g,\nu,t,p)) \mapsto (kg,\nu,t,p)$$

and the canonical momentum map J is

$$J\colon G \times \mathfrak{g}^* \times \mathbb{R}^2 \to \mathfrak{k}^*, \qquad (g, \nu, t, p) \mapsto (Coad_g \nu)_{|\mathfrak{k}}, \tag{5.1}$$

where $\mathfrak{k}$ denotes the Lie algebra of K.

On the other hand, we may consider the standard symplectic principal $\mathbb{R}$-bundle

$$\mu_\pi\colon T^*(G \times \mathbb{R}) \to V^*\pi$$

associated with the second projection $\pi\colon G \times \mathbb{R} \to \mathbb{R}$. Note that $V^*\pi$ is canonically identified with $T^*G \times \mathbb{R}$ or, in body coordinates, with $G \times \mathfrak{g}^* \times \mathbb{R}$. Then, $\mu_\pi\colon G \times \mathfrak{g}^* \times \mathbb{R}^2 \to G \times \mathfrak{g}^* \times \mathbb{R}$ is just the canonical projection and the corresponding principal $\mathbb{R}$-action is

$$\psi_\pi\colon \mathbb{R} \times (G \times \mathfrak{g}^* \times \mathbb{R}^2) \to G \times \mathfrak{g}^* \times \mathbb{R}^2, \qquad (s, (g, \nu, t, p)) \mapsto (g, \nu, t, p + s)$$

with infinitesimal generator $Z_{\mu_\pi} = \frac{\partial}{\partial p}$.

Now, we introduce a dynamics on the system. We will suppose that a time-dependent scalar product on $\mathfrak{g}$ is given or, equivalently, we will consider the corresponding (time-dependent) moment of inertia $\mathbb{I}_t\colon \mathfrak{g} \to \mathfrak{g}^*$. Moreover, we will suppose that a time-dependent potential function $V\colon G \times \mathbb{R} \to \mathbb{R}$ is given. Then, we will consider the homogeneous Hamiltonian function $F_h\colon G \times \mathfrak{g}^* \times \mathbb{R}^2 \to \mathbb{R}$ given by

$$F_h(g, \nu, t, p) = \frac{1}{2}\nu(\mathbb{I}_t^{-1}(\nu)) + V(g, t) + p, \tag{5.2}$$

for any $(g, \nu, t, p) \in G \times \mathfrak{g}^* \times \mathbb{R}^2$. Clearly, $Z_{\mu_\pi}(F_h) = 1$ and, thus, a Hamiltonian section $h\colon G \times \mathfrak{g}^* \times \mathbb{R} \to G \times \mathfrak{g}^* \times \mathbb{R}^2$ is induced. It is given by

$$h(g, \nu, t) = (g, \nu, t, -H(g, \nu, t)), \quad \text{for any } (g, \nu, t) \in G \times \mathfrak{g}^* \times \mathbb{R},$$

where

$$H(g, \nu, t) = \frac{1}{2}\nu(\mathbb{I}_t^{-1}(\nu)) + V(g, t).$$

Note that a scalar product on $\mathfrak{g}$ corresponds to a left invariant Riemannian metric on G and, consequently, to a left invariant fibred metric on the cotangent bundle $T^*G \simeq G \times \mathfrak{g}^*$. Such a kind of Hamiltonian sections (induced by a time-dependent Riemannian metric on the configuration manifold G and by a time-dependent potential function) are said to be *of mechanical type* and they have a particular importance in Mechanics, as we will see in the next examples.

Now, we will show a necessary and sufficient condition for h to be invariant with respect to the canonical action $T^*\phi$ on the symplectic principal $\mathbb{R}$-bundle μ_π.

Proposition 5.1. *Under the previous hypotheses, the Hamiltonian section h is K-equivariant if and only if V is ϕ-invariant.*

Proof. It's sufficient to recall that h is K-equivariant if and only if F_h is $T^*\phi$-invariant. □

Now, we will suppose that G admits a reductive decomposition with respect to the subgroup K, i.e. we will suppose that there exists an $Ad(K)$-invariant subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}.$$

Again, this is a natural hypothesis for a Lie group G, as we will see in Sections 5.2 and 5.3.

The following result shows that, under these hypotheses, a principal connection on the principal bundle $G \times \mathbb{R} \to G/K \times \mathbb{R}$ is given.

Proposition 5.2. *Let $\lambda\colon T(G \times \mathbb{R}) \to \mathfrak{k}$ be the $\mathfrak{k}$-valued 1-form given by*

$$\lambda(g, \xi, t, \dot{t}) = (Ad_g\xi)^{\mathfrak{k}}, \qquad \textit{for any } (g, \xi, t, \dot{t}) \in G \times \mathfrak{g} \times \mathbb{R}^2 \simeq T(G \times \mathbb{R}),$$

where $(Ad_g\xi)^{\mathfrak{k}}$ is the component of $Ad_g\xi$ along $\mathfrak{k}$. Then, λ is the connection 1-form associated to a principal connection on the principal bundle $\pi_{G,K}\colon G\times \mathbb{R} \to G/K \times \mathbb{R}$.

Proof. For any $\xi \in \mathfrak{k}$, consider the infinitesimal generator of the action ϕ given by

$$\xi_{G\times\mathbb{R}}(g, t) = (g, Ad_{g^{-1}}\xi, t, 0).$$

Thus, it's clear that $\lambda(\xi_{G\times\mathbb{R}}(g, t)) = \xi$, for any $\xi \in \mathfrak{k}$ and $(g, t) \in G \times \mathbb{R}$. Moreover, for any $(g, \xi) \in G \times \mathfrak{g}$ and $k \in K$, from the $Ad(K)$-equivariance of the subspace $\mathfrak{m}$, we have that

$$(Ad_k(Ad_g\xi))^{\mathfrak{k}} = Ad_k(Ad_g\xi)^{\mathfrak{k}}.$$

Consequently, for any $(g, \xi, t, \dot{t}) \in G \times \mathfrak{g} \times \mathbb{R}^2$ and $k \in K$, we get

$$\begin{aligned}\lambda((Tl)_k(g, \xi, t, \dot{t})) = \lambda(kg, \xi, t, \dot{t}) &= (Ad_k(Ad_g\xi))^{\mathfrak{k}} \\ &= Ad_k(Ad_g\xi)^{\mathfrak{k}} = Ad_k(\lambda(g, \xi, t, \dot{t})).\end{aligned}$$

Therefore, λ is G-equivariant with respect to the tangent lift action on $T(G\times \mathbb{R})$ and to the adjoint action on $\mathfrak{k}$. □

Fix $\nu \in \mathfrak{k}^*$. The connection 1-form λ allows to develop the standard symplectic principal $\mathbb{R}$-bundle reduction as in Section 3.3. Consider the following 1-form $\lambda_\nu = \langle \nu, \lambda \rangle$ on $G \times \mathbb{R}$ whose value at $(g,t) \in G \times \mathbb{R}$ is given by

$$\lambda_\nu(x)\colon T_{(g,t)}(G \times \mathbb{R}) \simeq \mathfrak{g} \times \mathbb{R} \to \mathbb{R}, \quad (\xi, \dot{t}) \mapsto \langle \widetilde{\nu}, Ad_g \xi \rangle,$$

where $\widetilde{\nu} \in \mathfrak{g}^*$ is the extension of ν to $\mathfrak{g}$ such that $\widetilde{\nu}_{|\mathfrak{k}} = \nu$ and $\widetilde{\nu}_{|\mathfrak{m}} = 0$.

The 1-form λ_ν may be interpreted as a map $\lambda_\nu\colon G \times \mathbb{R} \to T^*(G \times \mathbb{R})$ given by

$$\lambda_\nu(g,t) = (T_g^* l_{g^{-1}}(Ad_g^* \widetilde{\nu}), 0) \in T_g^* G \times T_t^* \mathbb{R} \simeq T_{(g,t)}^*(G \times \mathbb{R})$$

or, in body coordinates,

$$\lambda_\nu(g,t) = (g, Ad_g^* \widetilde{\nu}, t, 0).$$

The 1-form λ_ν is K-invariant and takes values in $J^{-1}(\nu)$, where J is the canonical momentum map (5.1).

Therefore, one may consider the 2-form β_{λ_ν} on $G/K_\nu \times \mathbb{R}$ obtained from $d\lambda_\nu$ by passing to the quotient with respect to the projection

$$\pi_{G \times \mathbb{R}, K_\nu}\colon G \times \mathbb{R} \to G/K_\nu \times \mathbb{R},$$

where K_ν is the isotropy group of ν in K. Now, we denote by B_{λ_ν} the 2-form on $T^*(G/K_\nu \times \mathbb{R})$ given by $B_{\lambda_\nu} = \pi^*_{G/K_\nu \times \mathbb{R}} \beta_{\lambda_\nu}$, where $\pi_{G/K_\nu \times \mathbb{R}}\colon T^*(G/K_\nu \times \mathbb{R}) \to G/K_\nu \times \mathbb{R}$ is the cotangent bundle projection.

Then, using Theorem 3.4, we obtain that the reduced symplectic principal $\mathbb{R}$-bundle

$$(\mu_\pi)_\nu\colon ((T^*(G \times \mathbb{R}))_\nu, (\Omega_{G \times \mathbb{R}})_\nu) \to (V^*\pi)_\nu$$

may be embedded in the standard symplectic principal bundle

$$\mu_{\pi^*_{1,\nu}}\colon (T^*(G/K_\nu \times \mathbb{R}), \Omega_{G/K_\nu \times \mathbb{R}} - B_{\lambda_\nu}) \to V^* \pi^*_{1,\nu}$$

associated with the second projection

$$\pi^*_{1,\nu}\colon G/K_\nu \times \mathbb{R} \to \mathbb{R}.$$

Note that the canonical symplectic 2-form on $T^*(G/K_\nu \times \mathbb{R})$ is deformed by the magnetic term B_{λ_ν}. In the next section, we will study this reduced symplectic principal bundle $\mu_{\pi^*_{1,\nu}}$ for two particular examples.

In what follows, we will suppose that $K = K_\nu$, so that the symplectic principal $\mathbb{R}$-bundles $(\mu_\pi)_\nu$ and $V^*\pi^*_{1,\nu}$ are isomorphic via the symplectic principal $\mathbb{R}$-bundle isomorphism

$$\varphi_{\lambda_\nu}\colon (T^*(G \times \mathbb{R}))_\nu \to T^*(G/K \times \mathbb{R})$$

given in Theorem 3.4. Modulo the identifications $T^*(G\times\mathbb{R})\simeq G\times\mathfrak{g}^*\times\mathbb{R}^2$ and $T^*(G/K\times\mathbb{R})\simeq T^*(G/K)\times\mathbb{R}^2$, the map φ_{λ_ν} is explicitely given by

$$(\varphi_{\lambda_\nu}[g,\alpha,t,p])(T_{(g,t)}\pi_{G,K}(v_g,\dot{t}))=(T_g^*l_{g^{-1}}(\alpha-Ad_g^*\widetilde{\nu}))(v_g)+\dot{t}p, \tag{5.3}$$

for any $[g,\alpha,t,p]\in(T^*(G\times\mathbb{R}))_\nu=J^{-1}(\nu)/K$, where $[g,\alpha,t,p]=\pi_\nu(g,\alpha,t,p)$ denotes the equivalence class of $(g,\alpha,t,p)\in T^*_{(g,t)}(G\times\mathbb{R})$.

Its inverse $\varphi_{\lambda_\nu}^{-1}$ may be explicitely computed and it is given by

$$\varphi_{\lambda_\nu}^{-1}(\bar{\alpha}_{[g]},t,p)=[g,T_e^*l_g(T_g^*\pi_{G,K}(\bar{\alpha}_{[g]}))+Ad_g^*\widetilde{\nu},t,p] \tag{5.4}$$

for any $(\bar{\alpha}_{[g]},t,p)\in T^*_{([g],t)}(G/K\times\mathbb{R})$, where $\pi_{G,K}\colon G\to G/K$ is the canonical projection.

Furthermore, if V is K-invariant, we may reduce the non-autonomous Hamiltonian system $(T^*(G\times\mathbb{R}),\mu_\pi,\Omega_{G\times\mathbb{R}},h)$. A natural question is the following one: under which hypotheses is the reduced Hamiltonian section h_ν again of mechanical type?

Let's compute the reduced Hamiltonian section $h_\nu\colon V^*\pi_1^*\to T^*(G/K\times\mathbb{R})$, using the symplectic principal $\mathbb{R}$-bundle isomorphism φ_{λ_ν}. In particular, we will compute the corresponding homogeneous Hamiltonian function $F_{h_\nu}\in C^\infty(T^*(G/K\times\mathbb{R}))$ which is characterized by

$$F_{h_\nu}\circ\varphi_{\lambda_\nu}\circ\pi_\nu=F_{h|J^{-1}(\nu)}. \tag{5.5}$$

Denote again by $[g,\alpha,t,p]\in(T^*(G\times\mathbb{R}))_\nu\simeq J^{-1}(\nu)/K$ the equivalence class of an element $(g,\alpha,t,p)\in J^{-1}(\nu)\subset T^*(G\times\mathbb{R})$ and by

$$\pi_{G,K}\colon G\to G/K$$

the canonical projection. Thus, using (5.2), (5.4) and (5.5), we get

$$\begin{aligned}F_{h_\nu}(\bar{\alpha}_{[g]},t,p)&=F_{h_\nu}\left(\varphi_{\lambda_\nu}\left(\varphi_{\lambda_\nu}^{-1}(\bar{\alpha}_{[g]},t,p)\right)\right)\\&=F_{h_\nu}(\varphi_{\lambda_\nu}[g,T_e^*l_g(T_g^*\pi_{G,K}(\bar{\alpha}_{[g]}))+Ad_g^*\widetilde{\nu},t,p])\\&=F_h\left(g,T_e^*l_g(T_g^*\pi_{G,K}(\bar{\alpha}_{[g]}))+Ad_g^*\widetilde{\nu},t,p\right)\\&=\frac{1}{2}\bar{\mathcal{G}}_{t,[g]}(\bar{\alpha}_{[g]},\bar{\alpha}_{[g]})+\frac{1}{2}Ad_g^*\widetilde{\nu}(\mathbb{I}_t^{-1}(Ad_g^*\widetilde{\nu}))\\&\qquad+\left(T_e^*l_g(T_g^*\pi_{G,K}(\bar{\alpha}_{[g]}))\right)(\mathbb{I}_t^{-1}(Ad_g^*\widetilde{\nu}))+\widetilde{V}([g],t)+p,\end{aligned} \tag{5.6}$$

where $\bar{\mathcal{G}}_{t,[g]}\colon T^*_{[g]}(G/K)\times T^*_{[g]}(G/K)\to\mathbb{R}$ is given by

$$\bar{\mathcal{G}}_{t,[g]}\left(\bar{\alpha}_{1[g]},\bar{\alpha}_{2[g]}\right)=\alpha_1\left(\mathbb{I}_t^{-1}(\alpha_2)\right)$$

with $\alpha_i = T_e^* l_g(T_g^* \pi_{G,K}(\bar{\alpha}_{i[g]})) \in \mathfrak{g}^*$, $i = 1, 2$ and $\widetilde{V}\colon G/K \times \mathbb{R} \to \mathbb{R}$ is the function obtained from the K-invariant time-dependent potential function $V\colon G \times \mathbb{R} \to \mathbb{R}$ by passing to the quotient.

Note that $\bar{\mathcal{G}}_{t,[g]}$ is bilinear, symmetric and positive definite. Therefore, $\bar{\mathcal{G}}_t\colon [g] \mapsto \bar{\mathcal{G}}_{t,[g]}$ is a bundle metric on the cotangent bundle $T^*(G/K)$ and corresponds to a Riemannian metric $\mathcal{G}_t$ on G/K.

However, the reduced Hamiltonian section h_ν is not in general of mechanical type. If $\nu = 0 \in \mathfrak{k}^*$, the corresponding element $\widetilde{\nu}$ is zero. As a consequence, we directly deduce the following result.

Proposition 5.3. *Under the previous hypotheses, if $\nu = 0 \in \mathfrak{k}^*$, then the reduced non-autonomous Hamiltonian system*

$$(T^*(G/K) \times \mathbb{R}^2, \mu_{\pi_1^*}, \Omega_{G/K\times\mathbb{R}} - B_{\lambda_\nu}, h_\nu)$$

is of mechanical type.

Finally, we conclude this section answering to the following natural question. Let $t \in \mathbb{R}$ and consider the G-invariant Riemannian metric whose value at $e \in G$ corresponds to $\mathbb{I}_t$. Clearly, by construction, this metric is invariant with respect to the left regular action $l\colon G \times G \to G$. On the other hand, a (right) action is induced on the quotient space G/K and it is given by

$$G/K \times G \to G/K, \qquad (Kg, g') \mapsto K(gg'),$$

where $Kg = [g] = \pi_{G,K} \in G/K$. We will denote by $\bar{r}_h\colon G/K \to G/K$ the partial map $\bar{r}_h[g] = [gh]$.

The following result gives a necessary and sufficient condition for the Riemannian metric $\mathcal{G}_t$ on G/K to be G-invariant with respect to this action.

Proposition 5.4. *Let $t \in \mathbb{R}$. Then, the Riemannian metric $\mathcal{G}_t$ on G/K is G-invariant if and only if the restriction to $\mathfrak{k}^o$ of the scalar product corresponding to $\mathbb{I}_t$ is $Ad^*(G)$-invariant, where $\mathfrak{k}^o$ is the annihilator of $\mathfrak{k}$ in $\mathfrak{g}^*$.*

Proof. First of all, we note that, if we denote by

$$\alpha = T_e^* l_g(T_g^* \pi_{G,K}(\bar{\alpha}_{[g]})), \quad \beta = T_e^* l_g(T_g^* \pi_{G,K}(\bar{\beta}_{[g]}))$$

with $\bar{\alpha}_{[g]}, \bar{\beta}_{[g]} \in T_{[g]}^*(G/K)$, then

$$\begin{aligned} T_e^* l_{gh}\left(T_{gh}^* \pi_{G,K}(T_{[gh]}^* \bar{r}_{h^{-1}}(\bar{\alpha}_{[g]}))\right) &= T_e^*(\bar{r}_{h^{-1}} \circ \pi_{G,K} \circ l_{gh})(\bar{\alpha}_{[g]}) \\ &= T_e^*(\pi_{G,K} \circ l_g \circ i_h)(\bar{\alpha}_{[g]}) = T_e^* i_h(\alpha) = Ad_h^* \alpha, \end{aligned}$$

where $i_h\colon G \to G$ is the inner automorphism. It follows that

$$\bar{\mathcal{G}}_{t,[gh]}\left(T^*_{[gh]}\bar{r}_{h^{-1}}(\bar{\alpha}_{[g]}), T^*_{[gh]}\bar{r}_{h^{-1}}(\bar{\beta}_{[g]})\right) = Ad^*_h\alpha(\mathbb{I}_t^{-1}(Ad^*_h(\beta))) \tag{5.7}$$

Now, let's prove that for any $g \in G$

$$T^*_e l_g(T^*_g\pi_{G,K}(T^*_{[g]}(G/K))) = (Ad_{g^{-1}}\mathfrak{k})^o. \tag{5.8}$$

Indeed, for any $\bar{\alpha}_{[g]} \in T^*_{[g]}(G/K)$, if $\xi \in \mathfrak{k}$, we get

$$\begin{aligned}
&\left(T^*_e l_g(T^*_g\pi_{G,K}(\bar{\alpha}_{[g]}))\right)(Ad_{g^{-1}}\xi)\\
&\quad = T^*_g\pi_{G,K}(\bar{\alpha}_{[g]})\left(T_e l_g(T_g l_{g^{-1}}(T_e r_g(\xi)))\right)\\
&\quad = \bar{\alpha}_{[g]}(T_g\pi_{G,K}(T_e r_g(\xi))) = 0,
\end{aligned}$$

where we have used the fact that $T_e r_g(\mathfrak{k}) = \ker T_g\pi_{G,K}$, for $g \in G$. Since the two spaces have the same dimension, the equality (5.8) holds.

As a consequence, using (5.7) and (5.8), we deduce that $\bar{\mathcal{G}}_t$ is G-invariant if and only if

$$\alpha(\mathbb{I}_t^{-1}(\beta)) = (Ad^*_h\alpha)\left(\mathbb{I}_t^{-1}(Ad^*_h\beta)\right), \quad \forall\, g, h \in G \;\; \forall\, \alpha, \beta \in (Ad_{g^{-1}}\mathfrak{k})^o. \tag{5.9}$$

Therefore, it's sufficient to prove that condition (5.9) is equivalent to

$$\alpha(\mathbb{I}_t^{-1}(\beta)) = (Ad^*_h\alpha)\left(\mathbb{I}_t^{-1}(Ad^*_h\beta)\right), \quad \forall\, h \in G \;\; \forall\, \alpha, \beta \in \mathfrak{k}^o. \tag{5.10}$$

Clearly, (5.10) is a particular case of (5.9), taking $g = e$. Conversely, if (5.10) holds, for any $g \in G$ and $\alpha, \beta \in (Ad_{g^{-1}}\mathfrak{k})^o$, we have that $Ad^*_{g^{-1}}\alpha, Ad^*_{g^{-1}}\beta \in \mathfrak{k}^o$. Then, using (5.10), for $g = h$, we have that

$$Ad^*_{g^{-1}}\alpha\left(\mathbb{I}_t^{-1}(Ad^*_{g^{-1}}\beta)\right) = \alpha(\mathbb{I}_t^{-1}(\beta)). \tag{5.11}$$

On the other hand, using (5.10) for $h = gk$, with $k \in G$, we get

$$\begin{aligned}
Ad^*_{g^{-1}}\alpha\left(\mathbb{I}_t^{-1}(Ad^*_{g^{-1}}\beta)\right) &= Ad^*_{gk}(Ad^*_{g^{-1}}\alpha)\left(\mathbb{I}_t^{-1}(Ad^*_{gk}(Ad^*_{g^{-1}}\beta))\right)\\
&= Ad^*_k\alpha\left(\mathbb{I}_t^{-1}(Ad^*_k\beta)\right).
\end{aligned} \tag{5.12}$$

Comparing (5.11) and (5.12), since $k \in G$ is arbitrary, we obtain the desired equivalence. □

5.2 The time-dependent heavy top

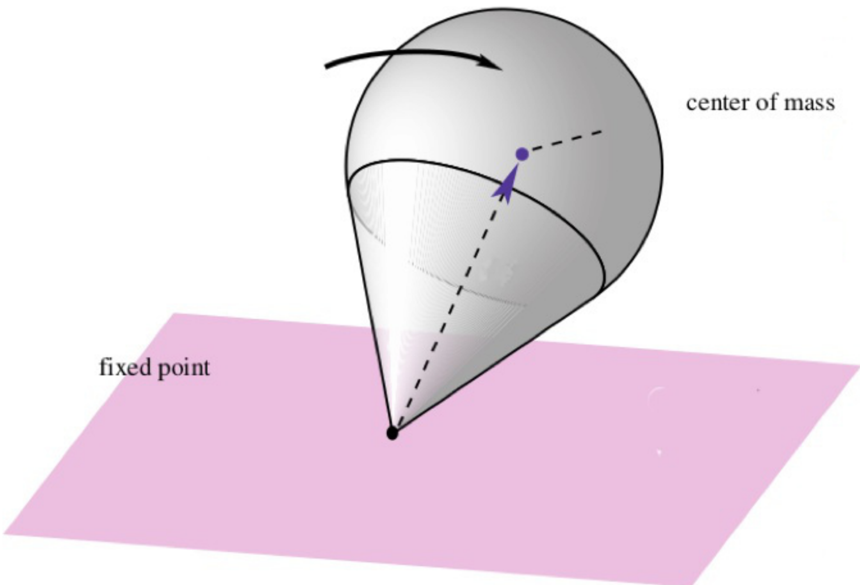
5.2.1 The reduced principal symplectic $\mathbb{R}$-bundle

This system consists of a rigid body with a fixed point moving in a time-dependent gravitational field (see [36] and references therein).

The configuration space for this mechanical system is the product manifold $SO(3)\times\mathbb{R}$ fibered on $\mathbb{R}$ by the second projection $\pi\colon SO(3)\times\mathbb{R}\to\mathbb{R}$. Moreover, the phase space of momenta $V^*\pi$ may be identified in a natural way with $(SO(3)\times\mathbb{R})\times\mathbb{R}^3$ using the left trivialization of $T^*SO(3)$. Under this identification, the Hamiltonian function $H\colon (SO(3)\times\mathbb{R})\times\mathbb{R}^3\to\mathbb{R}$ is given by

$$H((A,t),\Pi)=\frac{1}{2}\mathbb{I}^{-1}\Pi\cdot\Pi+A^{-1}e_3\cdot\gamma(t),$$

where $\mathbb{I}\colon \mathfrak{so}(3)\cong\mathbb{R}^3\to\mathfrak{so}^*(3)\cong\mathbb{R}^3$ is the inertial tensor of the body and $\gamma\colon\mathbb{R}\to\mathbb{R}^3$ is the time-dependent gravitational field. Here $\cdot\colon\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}$ is the canonical Euclidean scalar product on $\mathbb{R}^3$.



Now, we consider the closed subgroup of $SO(3)$

$$K=\left\{\begin{pmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\,:\,\theta\in\mathbb{R}\right\}\simeq S^1.$$

As in Section 1.4, we will denote by A_θ the matrix of rotation of an angle θ around the z-axis.

If $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathfrak{so}(3) \cong \mathbb{R}^3$, then the Lie algebra associated with K is just $\langle e_3 \rangle \simeq \mathbb{R}$.

In addition, as we have already seen, $SO(3)$ is a reductive Lie group with respect to K. So, we may consider a principal connection $\lambda\colon T(SO(3)\times\mathbb{R}) \to \mathbb{R}$ on the principal bundle

$$\pi_{SO(3)\times\mathbb{R},K}\colon SO(3) \times \mathbb{R} \to SO(3)/K \times \mathbb{R}.$$

as in Proposition 5.2. Recall that, modulo the identification $\mathfrak{so}(3) \simeq \mathbb{R}^3$, the adjoint action $Ad\colon SO(3) \times \mathbb{R}^3 \to \mathbb{R}^3$ of $SO(3)$ is just the standard action of $SO(3)$ on $\mathbb{R}^3$. Thus the principal connection $\lambda\colon T(SO(3) \times \mathbb{R}) \to \mathbb{R}$ is given by

$$\lambda(A, v, t, \dot{t}) = Av \cdot e_3.$$

Now, we consider the action of K on $T^*(SO(3) \times \mathbb{R}) \simeq (SO(3) \times \mathbb{R}) \times (\mathbb{R}^3 \times \mathbb{R})$ given by

$$\phi(A_\theta, ((A, t), \Pi, p)) = ((A_\theta A, t), \Pi, p),$$

with $A_\theta \in K$, $A \in SO(3)$, $t \in \mathbb{R}$ and $(\Pi, p) \in \mathbb{R}^3 \times \mathbb{R}$. This action is free and proper and π is invariant with respect to it.

In this section, we will develop the reduction procedure for the symplectc principal $\mathbb{R}$-bundle $\mu_\pi\colon (T^*(SO(3) \times \mathbb{R}), \Omega_{SO(3)\times\mathbb{R}}) \to V^*\pi$ with respect to the action of K at a level $\nu \in \mathbb{R}$.

Let $J\colon (SO(3) \times \mathbb{R}) \times (\mathbb{R}^3 \times \mathbb{R}) \to \mathbb{R}$ be the momentum map deduced from (1.50) whose explicit expression is

$$J((A, t), \Pi, p) = A\Pi \cdot e_3.$$

Now, for $\nu \in \mathbb{R}$, since $K \simeq S^1$ is abelian, the isotropic subgroup K_ν is just S^1. The level set $J^{-1}(\nu)$ is

$$\{\, ((A, t), (\Pi, p)) \in (SO(3) \times \mathbb{R}) \times (\mathbb{R}^3 \times \mathbb{R}) \mid A\Pi \cdot e_3 = \nu\}. \tag{5.13}$$

If we apply the cotangent bundle reduction using the principal connection $\lambda\colon T(SO(3) \times \mathbb{R}) \to \mathbb{R}$, we obtain that the reduced symplectic manifold $J^{-1}(\nu)/K$ is diffeomorphic to $T^*(S^2 \times \mathbb{R}) \cong T^*S^2 \times \mathbb{R}^2$.

In order to compute the explicit diffeomorphism between $J^{-1}(\nu)/K$ and $T^*S^2 \times \mathbb{R}^2$, we remark that the quotient space $SO(3)/K$ is canonically diffeomorphic to the 2-sphere S^2 via the diffeomorphism

$$[A] \in SO(3)/K \mapsto A^{-1}e_3 \in \mathbb{R}^3$$

This map is clearly well-defined and bijective. Note that the tangent map of the corresponding projection $SO(3) \to S^2$, $A \mapsto A^{-1}e_3$ is given by

$$(A, v) \in SO(3) \times \mathbb{R}^3 \simeq T(SO(3)) \mapsto (A^{-1}e_3, A^{-1}e_3 \times v) \in TS^2,$$

where we have identified $T_uS^2 \simeq \{w \in \mathbb{R}^3 \,|\, w \cdot u = 0\}$. Note that $A^{-1}e_3 \times v \in T_{A^{-1}e_3}S^2$ because $A^{-1}e_3 \cdot (A^{-1}e_3 \times v) = 0$.

Therefore, using (5.3) and (5.13), one may obtain that the map

$$\varphi_{\lambda_\nu} \colon J^{-1}(\nu)/K \to T^*S^2 \times \mathbb{R}^2$$

is explicitely given by

$$\varphi_{\lambda_\nu}([A, \Pi, t, p]) = (A^{-1}e_3, (A^{-1}e_3) \times \Pi, (t, p)), \tag{5.14}$$

where we have identified $T^*_uS^2$ with its dual $T_uS^2 \simeq \{w \in \mathbb{R}^3 \,|\, w \cdot u = 0\}$ via the standard metric on S^2.

On the other hand, recall that the map φ_{λ_ν} is a symplectomorphism if $T^*S^2 \times \mathbb{R}^2$ is equipped with the canonical symplectic 2-form on $T^*(S^2 \times \mathbb{R})$ deformed by the magnetic term B_{λ_ν} associated with λ_ν. This magnetic term may be written in terms of the symplectic area on S^2. Indeed, using the construction of the magnetic term given in Section 3.3, we get that B_{λ_ν} is just given by $\pi^*_{S^2\times\mathbb{R}}\beta_{\lambda_\nu}$, where $\pi_{S^2\times\mathbb{R}} \colon T^*(S^2 \times \mathbb{R}) \to S^2 \times \mathbb{R}$ is the canonical projection and β_{λ_ν} is the 2-form on $S^2 \times \mathbb{R}$ characterized by

$$\pi^*_{SO(3)\times\mathbb{R},K}\beta_{\lambda_\nu} = d\lambda_\nu. \tag{5.15}$$

Therefore, using (5.15), one may prove that $\beta_{\lambda_\nu} = -\nu\omega_{S^2}$, where ω_{S^2} is the symplectic area on S^2, i.e.

$$\omega_{S^2}(x)(u, v) = -x \cdot (u \times v), \quad \forall\, x \in S^2,\ u, v \in T_xS^2 \subseteq T_x\mathbb{R}^3 \cong \mathbb{R}^3.$$

As a consequence, we obtain that the reduced symplectic structure on $T^*S^2 \times \mathbb{R}^2$ is given by $\Omega_{S^2\times\mathbb{R}} + \nu\pi^*_{S^2\times\mathbb{R}}(\omega_{S^2})$, where $\Omega_{S^2\times\mathbb{R}}$ is the canonical symplectic 2-form on $T^*S^2 \times \mathbb{R}^2 \simeq T^*(S^2 \times \mathbb{R})$. We may resuming our results in the following

Theorem 5.5. *Let's consider the standard symplectic principal $\mathbb{R}$-bundle*

$$\mu_\pi \colon (T^*(SO(3)) \times \mathbb{R}, \Omega_{SO(3)\times\mathbb{R}}) \to V^*\pi$$

induced by the second projection $\pi \colon SO(3) \times \mathbb{R} \to \mathbb{R}$ and the closed Lie subgroup K of $SO(3)$ given by rotations around the z-axis.

Then, the symplectic principal $\mathbb{R}$*-bundle obtained by reducing* μ_π *with respect to the action of* K *at a level* $\nu \in \mathbb{R}$ *is isomorphic, via* φ_{λ_ν}*, to the standard symplectic principal bundle*

$$\mu_{\pi_1^*}\colon (T^*S^2 \times \mathbb{R}^2, \Omega_{S^2\times\mathbb{R}} + \nu\pi^*_{S^2\times\mathbb{R}}(\omega_{S^2})) \to T^*S^2 \times \mathbb{R}, \tag{5.16}$$

induced by the second projection $\pi_1^*\colon S^2 \times \mathbb{R} \to \mathbb{R}$*, where the canonical symplectic* 2*-form* $\Omega_{S^2\times\mathbb{R}}$ *on* $T^*S^2 \times \mathbb{R}$ *is deformed by the magnetic term* $\nu\pi^*_{S^2\times\mathbb{R}}(\omega_{S^2})$*.*

Using the results given in Section 3.4 (see Theorem 3.6), we may also obtain the following result

Corollary 5.6. *The Poisson bivector on* $T^*S^2 \times \mathbb{R}$ *induced on the base space of the symplectic principal* $\mathbb{R}$*-bundle* (5.16) *is given by*

$$\Lambda_{T^*S^2\times\mathbb{R}} - \nu\omega^v_{S^2},$$

where $\Lambda_{T^*S^2\times\mathbb{R}}$ *is the canonical Poisson bivector on* $T^*S^2 \times \mathbb{R}$ *and* $\omega^v_{S^2}$ *is the vertical lift to* $T^*S^2 \times \mathbb{R}$ *of the symplectic area* ω_{S^2} *on* S^2*.*

5.2.2 The reduced non-autonomous Hamiltonian system

Using the same notation as in Subsection 5.2.1, we have that the homogeneous Hamiltonian function $F_h\colon (SO(3)\times\mathbb{R})\times(\mathbb{R}^3\times\mathbb{R}) \to \mathbb{R}$ associated with this system is

$$F_h((A,t),\Pi,p) = \frac{1}{2}\langle \mathbb{I}^{-1}\Pi, \Pi\rangle + \langle A^{-1}e_3, \gamma(t)\rangle + p.$$

Now, we will apply the reduction process at a level $\nu \in \mathbb{R}$. The reduced homogenous Hamiltonian function $F_{h_\nu}\colon T^*S^2 \times \mathbb{R}^2 \to \mathbb{R}$ may be computed using (4.26) and Proposition 4.9 (referred to the symplectic principal $\mathbb{R}$-bundle isomorphism φ_{λ_ν}). Then, from (5.14), we obtain that

$$\begin{aligned} F_{h_\nu}(q,p_q,t,p) = {} & \frac{1}{2}\,\mathbb{I}^{-1}(p_q\times q)\cdot(p_q\times q) + \nu\,\mathbb{I}^{-1}(q)(p_q\times q) \\ & + \frac{1}{2}\,\nu^2\,(\mathbb{I}^{-1}q)\cdot q + q\cdot\gamma(t) + p \end{aligned}$$

for any $(q,p_q,t,p) \in T^*S^2 \times \mathbb{R}^2$, with $q \in S^2 \subset \mathbb{R}^3$ and $q\cdot p_q = 0$.

Notice that the function $F_{h_\nu}\colon T^*S^2\times\mathbb{R}^2\to\mathbb{R}$ is the restriction to $T^*S^2\times\mathbb{R}^2$ of $\widetilde{F}_{h_\nu}\colon\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^2\to\mathbb{R}$ given by

$$\begin{aligned}\widetilde{F}_{h_\nu}(q,p_q,t,p) &= \frac{1}{2}\,\mathbb{I}^{-1}(p_q\times q)\cdot(p_q\times q)+\nu\,\mathbb{I}^{-1}(q)(p_q\times q)\\ &\quad+\frac{1}{2}\,\nu^2\,(\mathbb{I}^{-1}q)\cdot q+q\cdot\gamma(t)+p\end{aligned}$$

Now, suppose that $\nu=0$, so that the symplectic 2-form on the total space $T^*S^2\times\mathbb{R}^2$ of the reduced symplectic principal $\mathbb{R}$-bundle is just the canonical symplectic 2-form $\Omega_{S^2\times\mathbb{R}}$.

The equations defining to T^*S^2 as a submanifold of $T^*\mathbb{R}^3\simeq\mathbb{R}^3\times\mathbb{R}^3$ are $\|q\|^2-1=0$ and $q\cdot p_q=0$. So, any extension of F_{h_0} has the form

$$\bar{F}_{h_0}(q,p_q,t,p)=\widetilde{F}_{h_0}(q,p_q,t,p)+\lambda(q\cdot p_q)+\mu(\|q\|^2-1),$$

where λ and μ are the Lagrange multipliers which we must determine. Then, since (q,p_q,t,p) are canonical coordinates for the corresponding symplectic principal $\mathbb{R}$-bundle, the Hamilton equations for this Hamiltonian function with initial condition on $T^*S^2\times\mathbb{R}^2$ are

$$\begin{cases}\dot{q}=q\times\mathbb{I}^{-1}(p_q\times q)+\lambda q\\ \dot{p_q}=p_q\times\mathbb{I}^{-1}(p_q\times q)-\gamma(t)-\lambda p_q-2\mu q\\ \dot{t}=1\\ \dot{p}=-q\cdot\dot{\gamma}(t)\end{cases}$$

with $(q,p_q)\in T^*S^2$. Since $q\cdot\dot{q}=0$ and $p_q\cdot\dot{q}+\dot{p_q}\cdot q=0$, then

$$\begin{cases}\dot{q}=q\times\mathbb{I}^{-1}(p_q\times q)\\ \dot{p_q}=p_q\times\mathbb{I}^{-1}(p_q\times q)-\gamma(t)+(q\cdot\gamma(t))q\\ \dot{t}=1\\ \dot{p}=-q\cdot\dot{\gamma}(t)\end{cases}$$

The projection of the solutions of these equations to $T^*S^2\times\mathbb{R}$ are just the solutions of the reduced dynamics, that is,

$$\begin{cases}\dot{q}=q\times\mathbb{I}^{-1}(p_q\times q)\\ \dot{p_q}=p_q\times\mathbb{I}^{-1}(p_q\times q)-\gamma(t)+(q\cdot\gamma(t))q\\ \dot{t}=1\end{cases}$$

In fact, the solutions of these equations are just the integral curves of the Reeb vector field associated with the cosymplectic manifold $T^*S^2 \times \mathbb{R}$ equipped with the cosymplectic structure $(\Omega_{S^2} + \mathrm{d}H_0 \wedge \mathrm{d}t, \mathrm{d}t)$ on $T^*S^2 \times \mathbb{R}$, where $H_0 \colon T^*S^2 \times \mathbb{R} \to \mathbb{R}$ is given by

$$H_0(q, p_q, t) = \frac{1}{2}\mathbb{I}^{-1}(p_q \times q) \cdot (p_q \times q) + q \cdot \gamma(t).$$

Note that if $\mathbb{I} = Id$ and $\gamma(t) = 0$ then the corresponding reduced Hamilton equations on T^*S^2 are

$$\dot{q} = p_q, \qquad \dot{p_q} = -\|\dot{q}\|^2 q,$$

which implies that $\ddot{q} = -\|\dot{q}\|^2 q$. Therefore, the projection to S^2 of the Hamilton equations are just geodesics of the standard metric of S^2.

5.3 The bidimensional time-dependent damped harmonic oscillator

5.3.1 The reduced symplectic principal $\mathbb{R}$-bundle

This time-dependent mechanical system involves harmonic oscillators with time-dependent frequency or with time-dependent masses or subject to linear time-dependent damping forces (see [9] and references therein).

The configuration space is the manifold $\mathbb{R}^2 \times \mathbb{R}$ fibered on $\mathbb{R}$ with respect to the surjective submersion $pr_2 \colon \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ and the corresponding restricted phase space of momenta is $V^*pr_2 = T^*\mathbb{R}^2 \times \mathbb{R}$. The Hamiltonian function $H \colon V^*pr_2 = T^*\mathbb{R}^2 \times \mathbb{R} \cong (\mathbb{R}^2 \times \mathbb{R}^2) \times \mathbb{R} \to \mathbb{R}$ is given by

$$H(q^1, q^2, p_1, p_2, t) = \frac{e^{\sigma(t)}}{2}(p_1^2 + p_2^2) + F(t)((q^1)^2 + (q^2)^2)$$

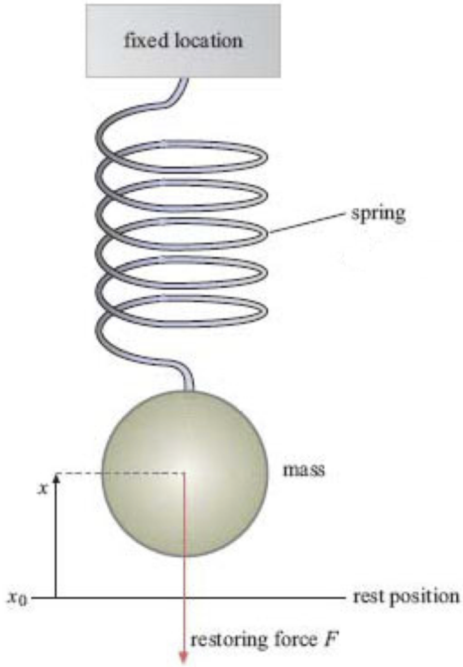
with $\sigma, F \colon \mathbb{R} \to \mathbb{R}$ real C^∞-functions on $\mathbb{R}$.

We consider the action $\phi \colon S^1 \times (\mathbb{R}^2 \times \mathbb{R}) \to (\mathbb{R}^2 \times \mathbb{R})$ of S^1 on $\mathbb{R}^2 \times \mathbb{R}$ given by

$$\phi_\theta(q^1, q^2, t) = (q^1 \cos\theta + q^2 \sin\theta, -q^1 \sin\theta + q^2 \cos\theta, t), \text{ for } \theta \in S^1$$

which is not free. However, if one restricts this action to $(\mathbb{R}^2 \setminus \{(0,0)\}) \times \mathbb{R} \cong (S^1 \times \mathbb{R}^+) \times \mathbb{R}$, ϕ is free and proper. Then, in order to reduce the system, we consider the second projection $\pi \colon (S^1 \times \mathbb{R}^+) \times \mathbb{R} \to \mathbb{R}$ and the corresponding symplectic $\mathbb{R}$-principal bundle

$$\mu_\pi \colon T^*(S^1 \times \mathbb{R}^+) \times \mathbb{R}^2 \to T^*(S^1 \times \mathbb{R}^+) \times \mathbb{R}, \quad (\theta, r, p_\theta, p_r, t, p) \mapsto (\theta, r, p_\theta, p_r, t).$$



A direct computation proves that $\pi \circ \phi_\theta = \pi$. Therefore, $T^*\phi\colon S^1 \times (T^*(S^1 \times \mathbb{R}^+) \times \mathbb{R}^2) \to T^*(S^1 \times \mathbb{R}^+) \times \mathbb{R}^2$ is a canonical action.

Note that the configuration space $S^1 \times \mathbb{R}^+$ is a Lie group and that the action ϕ (restricted to $\mathbb{R}^2 \setminus \{(0,0)\} \simeq S^1 \times \mathbb{R}^+$) is just the standard left action of S^1 on the first factor. In addition, since $S^1 \times \mathbb{R}^+$ is abelian, $S^1 \times \mathbb{R}^+$ obviously admits a reductive decomposition with respect to S^1. Thus, we may use the framework developed in Section 5.1 in order to reduce the symplectic principal $\mathbb{R}$-bundle μ_π with respect to the action of S^1.

In addition, we consider the momentum maps $J\colon T^*(S^1 \times \mathbb{R}^+) \times \mathbb{R}^2 \to \mathbb{R}$ and $J^{V^*\pi}\colon T^*(S^1 \times \mathbb{R}^+) \times \mathbb{R} \to \mathbb{R}$ deduced from (1.50) whose explicit expressions are

$$J(\theta, r, p_\theta, p_r, t, p) = p_\theta, \qquad J^{V^*\pi}(\theta, r, p_\theta, p_r, t) = p_\theta.$$

If $\nu \in \mathbb{R}$, then the corresponding level sets may be expressed as

$$J^{-1}(\nu) \cong S^1 \times T^*\mathbb{R}^+ \times \mathbb{R}^2, \qquad (J^{V^*\pi})^{-1}(\nu) \cong S^1 \times T^*\mathbb{R}^+ \times \mathbb{R}$$

and, since the isotropy subgroup of S^1 at ν is again S^1, the reduced spaces are just

$$J^{-1}(\nu)/S^1 \cong T^*(\mathbb{R}^+ \times \mathbb{R}), \qquad (J^{V^*\pi})^{-1}(\nu)/S^1 \cong T^*\mathbb{R}^+ \times \mathbb{R}.$$

Finally, the Poisson structure on $(J^{V^*\pi})^{-1}(\nu)/S^1 \cong T^*\mathbb{R}^+ \times \mathbb{R}$ is the one induced by the standard cosympletic structure.

5.3.2 The reduced non-autonomous Hamiltonian system

Using the same notation as in Subsection 5.3.1, we have that the homogeneous Hamiltonian function $F_h\colon T^*(S^1\times\mathbb{R}^+)\times\mathbb{R}^2\to\mathbb{R}$ is given by

$$F_h(\theta,r,p_\theta,p_r,t,p)=\frac{e^{\sigma(t)}}{2}(p_r^2+\frac{1}{r^2}p_\theta^2)+F(t)r^2+p.$$

Since F_h is $T^*\phi$-invariant, we may reduce the non-autonomous Hamiltonian system $(T^*(S^1\times\mathbb{R}^+)\times\mathbb{R}^2,\mu_\pi,\Omega,h)$ at $\nu\in\mathbb{R}$. The reduced homogeneous Hamiltonian system $F_{h_\nu}\colon T^*(\mathbb{R}^+\times\mathbb{R})\to\mathbb{R}$ is given by

$$F_{h_\nu}((r,t),(p_r,p))=\frac{e^{\sigma(t)}}{2}\left(p_r^2+\frac{\nu^2}{r^2}\right)+F(t)r^2+p.$$

Finally, the dynamics on the reduced system is described by the cosymplectic structure $(\omega_{h_\nu},\eta_{h_\nu})$ and the vector field $\mathcal{R}_{h_\nu}$ on $T^*\mathbb{R}^+\times\mathbb{R}$

$$\omega_{h_\nu}=dr\wedge dp_r+\left(2F(t)r-e^{\sigma(t)}\frac{\nu^2}{r^3}\right)dr\wedge dt+e^{\sigma(t)}p_r dp_r\wedge dt,\qquad \eta_{h_\nu}=dt,$$

$$\mathcal{R}_{h_\nu}=\frac{\partial}{\partial t}+e^{\sigma(t)}p_r\frac{\partial}{\partial r}+\left(e^{\sigma(t)}\frac{\nu^2}{r^3}-2F(t)r\right)\frac{\partial}{\partial p_r}.$$

5.4 The frame-independent formulation of the analytical mechanics in a Newtonian space-time

The *Newtonian space-time* is a system (E,τ,g) where E is an affine space modelled over the n-dimensional vector space V, τ is a non-zero element of V^* and $g\colon V_0\to V_0^*$ is a scalar product on $V_0=\ker\tau$. Let V_1 be the affine subspace of V defined by the equation $\tau(v)=1$. An element of V_1 may be interpreted as the family of inertial observers that move in the space-time with the constant velocity u (see [22]). We will denote by $i\colon V_0\to V$ the inclusion and by $g'=i\circ g^{-1}\circ i^*\colon V^*\to V$ the contravariant tensor on V defined by g.

If u is a fixed inertial frame, the homogeneous Hamiltonian function on $T^*E\simeq E\times V^*$ is given by

$$H_u(x,\alpha)=\alpha(u)+\frac{1}{2m}\alpha(g'(\alpha))+\varphi(x),\qquad \text{for any } x\in E,\ \alpha\in V^*,$$

where $\varphi\colon E \to \mathbb{R}$ is the potential.

Note that the expression of the homogeneous Hamiltonian function *depends* on the inertial frame. In the mechanics of non-autonomous systems, we are forced to choose a reference frame on the space-time (see [22]). The aim of this section is to describe a frame-independent formulation of the dynamics and describe how a reduction procedure may be applied in the symplectic principal $\mathbb{R}$-bundle setting.

In order to do this, we collect all the homogeneous Hamiltonian functions for all inertial frames and construct for them a universal object which does not depend on an inertial frame. We consider the set $V_1 \times E \times V^*$ which contains all the information about the reference frame and the momentum of the system. Then, we may introduce the relation on $V_1 \times E \times V^*$ given by

$$(u, x, \alpha) \sim (u', x', \alpha') \quad \Leftrightarrow \quad x = x' \text{ and } H_u(x, \alpha) = H_{u'}(x', \alpha')$$

for any $(u, x, \alpha), (u', x', \alpha') \in V_1 \times E \times V^*$. Equivalently, if $\sigma\colon V_1 \times V_1 \to V^*$ is the map defined by

$$\sigma(u, u')(v) = g(u - u')\left(v - \tau(v)\frac{u + u'}{2}\right), \qquad v \in V,$$

then

$$(u, x, \alpha) \sim (u', x', \alpha') \quad \Leftrightarrow \quad x = x' \text{ and } \alpha' = \alpha + m\sigma(u, u').$$

A straightforward computation shows that the map $\sigma\colon V_1 \times V_1 \to V^*$ verifies the following properties:

$$\begin{aligned} \sigma(u, u) &= 0, \\ \sigma(u, u') &= -\sigma(u', u), \\ \sigma(u, u') + \sigma(u', u'') &= \sigma(u, u''), \end{aligned}$$

for any $u, u', u'' \in V_1$. Thus, $\sim$ is an equivalence relation.

The quotient space $P = (V_1 \times E \times V^*)/\sim$ is an affine bundle over E modelled over the vector bundle $E \times V^* \to E$. The affine action $+\colon P \times_E (E \times V^*) \to P$ is given by

$$[u, x, \alpha] + (x, \beta) = [u, x, \alpha + \beta],$$

for any $[u, x, \alpha] \in P$ and $(x, \beta) \in E \times V^*$.

Moreover, if $u \in V_1$ is fixed, there exists a unique symplectic form Ω on P such that the map $\Theta_u\colon T^*E \cong E \times V^* \to P$ given by $\Theta_u(x, \gamma) = [(u, x, \gamma)]$ is a symplectomorphism, where $E \times V^* \simeq T^*E$ is equipped with the canonical symplectic 2-form. If $u' \in V_1$ then, since $\Theta_u^{-1} \circ \Theta_{u'} \colon T^*E \to T^*E$ is just the translation by the constant 1-form $\sigma(u', u)$, Ω doesn't depend on the chosen inertial frame u.

On the other hand, we may consider the action $\psi \colon \mathbb{R} \times P \to P$ given by

$$\psi(s, [u, x, \alpha]) = [u, x, \alpha + s\tau], \qquad \text{for any } s \in \mathbb{R} \text{ and } [u, x, \alpha] \in P. \tag{5.17}$$

In what follows, we will show that P is the total space of a symplectic principal $\mathbb{R}$-bundle with corresponding principal $\mathbb{R}$-action $\psi \colon \mathbb{R} \times P \to P$.

Consider the equivalence relation on $V_1 \times E \times V_0^*$ given by

$$(u, x, \bar{\alpha}) \sim_0 (u', x', \bar{\alpha}') \quad \Leftrightarrow \quad x = x' \text{ and } \bar{\alpha}' = \bar{\alpha} + mg(u - u'),$$

for any $(u, x, \bar{\alpha}), (u', x', \bar{\alpha}') \in V_1 \times E \times V_0^*$. Let P_0 be the quotient space $(V_1 \times E \times V_0^*)/\sim_0$ and $\mu \colon P \to P_0$ the map given by

$$\mu[u, x, \alpha] = [u, x, \alpha_{|V_0}], \tag{5.18}$$

for any $[u, x, \alpha] \in P = (V_1 \times E \times V^*)/\sim$.

Theorem 5.7. *Under the previous hypotheses, the map $\mu \colon (P, \Omega) \to P_0$ given by (5.18) is a symplectic principal $\mathbb{R}$-bundle with corresponding principal $\mathbb{R}$-action $\psi \colon \mathbb{R} \times P \to P$ given by (5.17).*

Proof. If we fix an inertial frame $u \in V_1$, we may use the symplectomorphism $\Theta_u \colon T^*E \simeq E \times V^* \to P$ in order to obtain a corresponding action $\bar{\psi}$ on T^*E. Then, one may easily prove that $\bar{\psi} \colon \mathbb{R} \times T^*E \to T^*E$ is just given by

$$\bar{\psi}_s(x, \alpha) = (x, \alpha + s\tau), \qquad \text{for any } s \in \mathbb{R} \text{ and } (x, \alpha) \in E \times V^* \simeq T^*E.$$

Thus, $\bar{\psi}$ (and, then, ψ) is free, proper and symplectic.

Finally, one may easily prove that $P/\mathbb{R}$ is diffeomorphic to P_0 and that the projection corresponding to the action ψ is just μ. $\square$

Now, we consider the following Hamiltonian section $h \colon P_0 \to P$

$$h[u, x, \bar{\alpha}] = \left[u, x, \bar{\alpha} \circ i_u - \left(\frac{1}{2m}\bar{\alpha}(g^{-1}(\bar{\alpha})) + \varphi(x)\right)\tau\right],$$

where $i_u \colon V \to V_0$ is the projection $v \mapsto v - \tau(v)u$. The quadruple (P, μ, Ω, h) is said to be the *frame-independent dynamical system associated with the non-autonomous Hamiltonian system in the Newtonian space-time.*

Let's compute the homogeneous Hamiltonian function $F_h\colon P \to \mathbb{R}$ corresponding to h. From (4.1), we have that for any $[u, x, \alpha]$

$$\begin{aligned}
[u, x, \alpha] &= \psi(F_h[u, x, \alpha], h(\mu[u, x, \alpha])) \\
&= \left[u, x, \bar{\alpha} \circ i_u - \left(\frac{1}{2m}\bar{\alpha}(g^{-1}(\bar{\alpha})) + \varphi(x) - F_h[u, x, \alpha]\right)\tau\right],
\end{aligned}$$

where $\bar{\alpha} = \alpha_{|V_0}$, for any $[u, x, \alpha] \in P$. As a consequence,

$$\alpha = \bar{\alpha} \circ i_u - \left(\frac{1}{2m}\bar{\alpha}(g^{-1}(\bar{\alpha})) + \varphi(x) - F_h[u, x, \alpha]\right)\tau$$

and, if we compute both sides on $u \in V_1$, we obtain that

$$\alpha(u) = -\frac{1}{2m}\bar{\alpha}(g^{-1}(\bar{\alpha})) - \varphi(x) + F_h[u, x, \alpha].$$

Thus, the homogeneous Hamiltonian function $F_h\colon P \to \mathbb{R}$ corresponding to h is just given by

$$F_h[u, x, \alpha] = H_u(x, \alpha)$$

and doesn't depend on the chosen inertial frame u by construction.

Now, we will introduce a symmetry in the system: let G a subgroup of the group of the affine transformations of A. Suppose that for any $f_L \in G$, where $L\colon V \to V$ is the corresponding linear map, we have that

$$L^*\tau = \tau, \quad L_{|V_0} \text{ preserves } g \text{ and } \varphi \circ f_L = \varphi.$$

Moreover, we will suppose that $\mathfrak{g} \subset Aff(E, V_0)$. Then, we may consider the action $\phi\colon G \times P \to P$ defined by

$$(f_L, [u, x, \alpha]) \mapsto [Lu, f_L(x), (L^{-1})^*\alpha].$$

A straightforward computation shows that ϕ is a canonical action on the symplectic principal $\mathbb{R}$-bundle $\mu\colon P \to P_0$. Finally, we will suppose that the induced action $\phi^{P_0}\colon G \times P_0 \to P_0$ is free and proper. If not, one may restrict to a subset of E (probably, an open subset) and repeat the proofs. For any reference frame $u \in V_1$, one may consider the momentum map $J_u\colon P \to \mathfrak{g}^*$ defined by

$$J_u([w, x, \alpha]) = (\alpha - m\sigma(u, w))\,(\xi_E(x)), \qquad \text{for any } [w, x, \alpha] \in P \text{ and } \xi \in \mathfrak{g},$$

where $\xi_E \in \mathfrak{X}(E)$ is the infinitesimal generator associated with ξ for the natural action of G on E and we are identifying $T_xE \simeq V$. Then, one obtain the following result

Theorem 5.8. *Under the previous hypotheses, if* $\nu \in \mathfrak{g}^*$ *and* $u \in V_1$ *are fixed, a reduced symplectic principal* $\mathbb{R}$*-bundle* $\mu_\nu\colon (P_\nu, \Omega_\nu) \to (P_0)_\nu$ *and a reduced Hamiltonian section* $h_\nu\colon (P_0)_\nu \to P_\nu$ *are given, where*

$$P_\nu = J_u^{-1}(\nu)/G_\nu, \qquad (P_0)_\nu = (J_u^{P_0})^{-1}(\nu)/G_\nu.$$

Here, $J_u^{P_0}\colon P_0 \to \mathfrak{g}^*$ is the momentum map for the Poisson action of G on $P_0 = (V_1 \times E \times V_0^*)/\sim_0$.

Conclusions and future work

In this thesis, we have introduced the notion of a symplectic principal $\mathbb{R}$-bundle. The total space of a symplectic principal $\mathbb{R}$-bundle is a symplectic manifold and the principal action is symplectic. As a consequence, we deduce that the base space is a Poisson manifold of corank 1. Symplectic principal $\mathbb{R}$-bundles play an important role in the geometric description of non-autonomous Hamiltonian systems. In particular, we have developed a reduction procedure for such a kind of structures. We have proved a version of the Kirillov-Kostant-Souriau theorem (on the symplectic structure of the coadjoint orbits) in the setting of symplectic principal $\mathbb{R}$-bundles. In fact, from a principal G-bundle over the real line with total space M, a standard symplectic principal $\mathbb{R}$-bundle may be constructed and our reduction process may be applied. As a consequence, we deduce that the space of orbits M/G_ν of the action of G_ν on M admits a canonical Poisson structure of corank 1 whose symplectic leaves are isomorphic to the coadjoint orbit which contains the element $\nu \in \mathfrak{g}^*$. Here, $\mathfrak{g}$ is the Lie algebra of G and G_ν is the isotropy group of ν with respect to the coadjoint action of G on $\mathfrak{g}^*$.

From a mechanical point of view, the reduction procedure allows to eliminate the symmetries of a G-invariant non-autonomous mechanical system (represented by a symplectic principal $\mathbb{R}$-bundle and an equivariant Hamiltonian section on it) and obtain a new mechanical system with a lesser number of degrees of freedom.

We discuss the case of the standard symplectic principal $\mathbb{R}$-bundle associated with a fibration over the real line. In such a case, under suitable hypotheses, the reduced symplectic principal $\mathbb{R}$-bundle is again standard, but some magnetic deformations appear. The canonical symplectic 2-form on the cotangent bundle is deformed by a magnetic term as in the classic cotangent bundle reduction, while the canonical Poisson bivector on the reduced base space is deformed by a corresponding (contravariant) magnetic term.

Finally, we have presented some examples which may be treated using our framework.

Along the thesis, we assume the regularity of the canonical action on the symplectic principal $\mathbb{R}$-bundle. Then, one may ask if a similar construction holds if we relax this assumption in order to include other examples (see, for instance, Example 5.3.1). We expect that well-known methods on singular reduction (see [3, 6, 53, 54, 55]) could be applied in the time-dependent setting. We remark that this reduction process could be not functorial, in the sense that a more general object (as, for instance, a suitable map between stratified spaces) could be needed.

On the other hand, suppose that a canonical action of a connected Lie group G on a symplectic principal $\mathbb{R}$-bundle is given and that G has a closed normal subgroup K. Then one could obtain a reduced symplectic principal $\mathbb{R}$-bundle in a two step procedure: first reducing by K and then by an appropriate quotient group. It would be interesting to discuss this procedure which is called *reduction by stages* (for reduction by stages in the symplectic framework, see [42]).

It is well-known that the phase space of velocities of a time-independent mechanical system is the tangent bundle TM of the configuration space M. A reduction theory for time-independent Lagrangian systems, which is the analogue of symplectic reduction, has been developed by several authors [28, 46, 47, 48].

On the other hand, in Section 1.3, we have seen that the 1-jet bundle $J^1\pi$ associated with a fibration $\pi\colon M \to \mathbb{R}$ is the phase space for the Lagrangian formulation of time-dependent mechanical systems. In fact, the Lagrangian function is a real function on $J^1\pi$. So, it would be interesting to develop a reduction process for 1-jet bundles associated with fibrations over the real line in the presence of a symmetry group. This will provide a Lagrangian parallel to the theory of reduction of symplectic principal $\mathbb{R}$-bundles which has been developed in this thesis.

Another important geometric object, which is widely used for reduction purposes, is a *Lie algebroid*. A Lie algebroid is a natural extension of the tangent bundle of a manifold and a real Lie algebra of finite dimension. In fact, a Lie algebroid over a manifold M is a vector bundle $\tau\colon A \to M$ such that the space of sections $\Gamma(A)$ admits a Lie algebra bracket $[\cdot,\cdot]$ and there exists a bundle morphism $\rho\colon A \to TM$ such that, if we denote again by $\rho\colon \Gamma(A) \to \mathfrak{X}(M)$ the corresponding morphism of $C^\infty(M)$-modules, the following condition (*Leibniz rule*) is satisfied

$$[X, fY] = \rho(X)(f)Y + f[X, Y],$$

for any $X, Y \in \Gamma(A)$ and $f \in C^\infty(M)$ (see [38, 56]).

There exists a one-to-one correspondence between Lie algebroid structures on a real vector bundle $A \to M$ and linear Poisson structures on the dual bundle $A^* \to M$ to A (see [11, 12]).

This is interesting for dynamical purposes. Indeed, standard formulation of Lagrangian and Hamiltonian Mechanics may be extended to Lie algebroids (see [13, 50]). The main feature of the Lie algebroid framework is its inclusive nature. Under the same umbrella, one can consider several situations as system with symmetries, systems evolving on semidirect products or Lagrangian and Hamiltonian systems on Lie algebras (see [10, 13]).

Very recently, in [40], the authors extend the Marsden-Weinstein reduction theorem for symplectic manifolds to the so-called *symplectic Lie algebroids*. The typical example of a symplectic Lie algebroid is the A-tangent bundle to A^*, for an arbitrary Lie algebroid A. It plays the same role in the symplectic Lie algebroid setting that the cotangent bundle in symplectic geometry. Indeed, if A is the standard Lie algebroid $TQ \to Q$, then the A-tangent bundle to A^* is just the standard Lie algebroid $T(T^*Q) \to T^*Q$. In [40], the authors discuss an embedding version of the reduction of A-tangent bundle to A^*.

On the other hand, an affine version of the notion of a Lie algebroid was introduced in [19, 51] (namely, a *Lie affgebroid*) and applications to time-dependent Mechanics were discussed in [20, 27, 51, 57]. Indeed the standard example of a Lie affgebroid is the 1-jet bundle associated with a fibration over the real line. Lie affgebroid structures on an affine bundle $A \to M$ are in one-to-one correspondence with Jacobi algebroid structures on the bidual bundle to A. A Jacobi algebroid structure on a real vector bundle is a Lie algebroid structure plus a 1-cocycle in the cohomology complex of the Lie algebroid. So, the definition of a symplectic Lie affgebroid may be introduced in a natural way and it would be interesting to extend the results in [40] for symplectic Lie affgebroids.

A bundle version of the reduction of symplectic Lie algebroids has not been developed. For this reason, a future research will be oriented to describe such a version and to extend these result to symplectic Lie affgebroids.

Bibliography

[1] R. Abraham and J.E. Marsden. *Foundations of Mechanics*. Benjamin/Cummings Publishing Co. Inc. Advanced Book Program, Reading, Mass., 1978. Second edition, revised and enlarged, with the assistance of Tudor Ratiu and Richard Cushman.

[2] C. Albert. Le théorème de réduction de Marsden-Weinstein en géométrie cosymplectique et de contact. *J. Geom. Phys.*, 6(4):627–649, 1989.

[3] J. M. Arms, R. H. Cushman, and M. J. Gotay. A universal reduction procedure for Hamiltonian group actions. In *The geometry of Hamiltonian systems (Berkeley, CA, 1989)*, volume 22 of *Math. Sci. Res. Inst. Publ.*, pages 33–51. Springer, New York, 1991.

[4] V. Arnold. Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. *Ann. Inst. Fourier (Grenoble)*, 16(fasc. 1):319–361, 1966.

[5] V. I. Arnold. *Mathematical methods of classical mechanics*. Springer-Verlag, New York, 1978. Translated from the Russian by K. Vogtmann and A. Weinstein, Graduate Texts in Mathematics, 60.

[6] L. Bates and E. Lerman. Proper group actions and symplectic stratified spaces. *Pacific J. Math.*, 181(2):201–229, 1997.

[7] K. H. Bhaskara and K. Viswanath. *Poisson algebras and Poisson manifolds*, volume 174 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow, 1988.

[8] F. Cantrijn, M. de León, and E. A. Lacomba. Gradient vector fields on cosymplectic manifolds. *J. Phys. A*, 25(1):175–188, 1992.

[9] José F. Cariñena, Javier de Lucas, and Manuel F. Rañada. Lie systems and integrability conditions for t-dependent frequency harmonic oscillators. *Int. J. Geom. Methods Mod. Phys.*, 7(2):289–310, 2010.

[10] J. Cortés, M. de León, J. C. Marrero, D. Martín de Diego, and E. Martínez. A survey of Lagrangian mechanics and control on Lie algebroids and groupoids. *Int. J. Geom. Methods Mod. Phys.*, 3(3):509–558, 2006.

[11] A. Coste, P. Dazord, and A. Weinstein. Groupoïdes symplectiques. In *Publications du Département de Mathématiques. Nouvelle Série. A, Vol. 2*, volume 87 of *Publ. Dép. Math. Nouvelle Sér. A*, pages i–ii, 1–62. Univ. Claude-Bernard, Lyon, 1987.

[12] T. J. Courant. Dirac manifolds. *Trans. Amer. Math. Soc.*, 319(2):631–661, 1990.

[13] M. de León, J. C. Marrero, and E. Martínez. Lagrangian submanifolds and dynamics on Lie algebroids. *J. Phys. A*, 38(24):R241–R308, 2005.

[14] de León, M., J. Marín-Solano, and J. C. Marrero. The constraint algorithm in the jet formalism. *Differential Geom. Appl.*, 6(3):275–300, 1996.

[15] de León, M., J. C. Marrero, and D. Martín de Diego. Time-dependent constrained Hamiltionian systems and Dirac brackets. *J. Phys. A*, 29(21):6843–6859, 1996.

[16] de León, M. and P. R. Rodrigues. *Methods of differential geometry in analytical mechanics*, volume 158 of *North-Holland Mathematics Studies.* North-Holland Publishing Co., Amsterdam, 1989.

[17] L. Di Terlizzi and J. J. Konderak. Reduction theorems for manifolds with degenerate 2-form. *J. Lie Theory*, 17(3):563–581, 2007.

[18] H. Geiges. Constructions of contact manifolds. *Math. Proc. Cambridge Philos. Soc.*, 121(3):455–464, 1997.

[19] K. Grabowska, J. Grabowski, and P. Urbański. Lie brackets on affine bundles. *Ann. Global Anal. Geom.*, 24(2):101–130, 2003.

[20] K. Grabowska, J. Grabowski, and P. Urbański. AV-differential geometry: Poisson and Jacobi structures. *J. Geom. Phys.*, 52(4):398–446, 2004.

[21] K. Grabowska, J. Grabowski, and P. Urbański. AV-differential geometry: Euler-Lagrange equations. *J. Geom. Phys.*, 57(10):1984–1998, 2007.

[22] K. Grabowska and P. Urbański. AV-differential geometry and Newtonian mechanics. *Rep. Math. Phys.*, 58(1):21–40, 2006.

[23] K. Grabowska and P. Urbański. AV-differential geometry and calculus of variations. In *XV International Workshop on Geometry and Physics*, volume 11 of *Publ. R. Soc. Mat. Esp.*, pages 144–152. R. Soc. Mat. Esp., Madrid, 2007.

[24] Katarzyna Grabowska, Janusz Grabowski, and Paweł Urbański. Lie brackets on affine bundles. *Ann. Global Anal. Geom.*, 24(2):101–130, 2003.

[25] V. Guillemin, E. Lerman, and S. Sternberg. *Symplectic fibrations and multiplicity diagrams.* Cambridge University Press, Cambridge, 1996.

[26] N. J. Hitchin, A. Karlhede, U. Lindström, and M. Roček. Hyper-Kähler metrics and supersymmetry. *Comm. Math. Phys.*, 108(4):535–589, 1987.

[27] D. Iglesias, J. C. Marrero, E. Padrón, and D. Sosa. Lagrangian submanifolds and dynamics on Lie affgebroids. *Rep. Math. Phys.*, 57(3):385–436, 2006.

[28] S. M. Jalnapurkar and J. E. Marsden. Reduction of Hamilton's variational principle. *Dyn. Stab. Syst.*, 15(3):287–318, 2000.

[29] D. Kazhdan, B. Kostant, and S. Sternberg. Hamiltonian group actions and dynamical systems of Calogero type. *Comm. Pure Appl. Math.*, 31(4):481–507, 1978.

[30] A. A. Kirillov. *Elements of the theory of representations.* Springer-Verlag, Berlin, 1976. Translated from the Russian by Edwin Hewitt, Grundlehren der Mathematischen Wissenschaften, Band 220.

[31] J. Klein. Espaces variationnels et mécanique. *Ann. Inst. Fourier (Grenoble)*, 12:1–124, 1962.

[32] S. Kobayashi and K. Nomizu. *Foundations of differential geometry. Vol I.* Interscience Publishers, a division of John Wiley & Sons, New York-Lond on, 1963.

[33] B. Kostant. On certain unitary representations which arise from a quantization theory. In *Group Representations in Math. and Phys. (Battelle Seattle 1969 Rencontres)*, Lecture Notes in Physics, Vol. 6, pages 237–253. Springer, Berlin, 1970.

[34] M. Kummer. On the construction of the reduced phase space of a Hamiltonian system with symmetry. *Indiana Univ. Math. J.*, 30(2):281–291, 1981.

[35] I Lacirasella, J.C. Marrero, and E. Padrón. Reduction of symplectic principal $\mathbb{R}$-bundles. *J. Phys. A*, 45(32):325202, 2012.

[36] D. Lewis, T. Ratiu, J. C. Simo, and J. E. Marsden. The heavy top: a geometric treatment. *Nonlinearity*, 5(1):1–48, 1992.

[37] P. Libermann and C. M. Marle. *Symplectic geometry and analytical mechanics*, volume 35 of *Mathematics and its Applications*. D. Reidel Publishing Co., Dordrecht, 1987. Translated from the French by Bertram Eugene Schwarzbach.

[38] K. Mackenzie. *Lie groupoids and Lie algebroids in differential geometry*, volume 124 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1987.

[39] G.-M. Marle. Symplectic manifolds, dynamical groups, and Hamiltonian mechanics. In *Differential geometry and relativity*, pages 249–269. Mathematical Phys. and Appl. Math., Vol. 3. Reidel, Dordrecht, 1976.

[40] J. C. Marrero, E. Padrón, and M. Rodríguez-Olmos. Reduction of a symplectic-like Lie algebroid with momentum map and its application to fiberwise linear Poisson structures. *J. Phys. A*, 45(16):165201, 34, 2012.

[41] J. Marsden, R. Montgomery, and T. Ratiu. Reduction, symmetry, and phases in mechanics. *Mem. Amer. Math. Soc.*, 88(436):iv+110, 1990.

[42] J. E. Marsden, G. Misiołek, J. P. Ortega, M. Perlmutter, and T. Ratiu. *Hamiltonian reduction by stages*, volume 1913 of *Lecture Notes in Mathematics*. Springer, Berlin, 2007.

[43] J. E. Marsden and M. Perlmutter. The orbit bundle picture of cotangent bundle reduction. *C. R. Math. Acad. Sci. Soc. R. Can.*, 22(2):35–54, 2000.

[44] J. E. Marsden and T. Ratiu. Reduction of Poisson manifolds. *Lett. Math. Phys.*, 11(2):161–169, 1986.

[45] J. E. Marsden and T. S. Ratiu. *Introduction to mechanics and symmetry*, volume 17 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 1994. A basic exposition of classical mechanical systems.

[46] J. E. Marsden, T. S. Ratiu, and J. Scheurle. Reduction theory and the Lagrange-Routh equations. *J. Math. Phys.*, 41(6):3379–3429, 2000.

[47] J. E. Marsden and J. Scheurle. Lagrangian reduction and the double spherical pendulum. *Z. Angew. Math. Phys.*, 44(1):17–43, 1993.

[48] J. E. Marsden and J. Scheurle. The reduced Euler-Lagrange equations. In *Dynamics and control of mechanical systems (Waterloo, ON, 1992)*, volume 1 of *Fields Inst. Commun.*, pages 139–164. Amer. Math. Soc., Providence, RI, 1993.

[49] J. E. Marsden and A. Weinstein. Reduction of symplectic manifolds with symmetry. *Rep. Mathematical Phys.*, 5(1):121–130, 1974.

[50] E. Martínez. Lagrangian mechanics on Lie algebroids. *Acta Appl. Math.*, 67(3):295–320, 2001.

[51] E. Martínez, T. Mestdag, and W. Sarlet. Lie algebroid structures and Lagrangian systems on affine bundles. *J. Geom. Phys.*, 44(1):70–95, 2002.

[52] B. O'Neill. *Semi-Riemannian geometry*, volume 103 of *Pure and Applied Mathematics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1983. With applications to relativity.

[53] J. P. Ortega and T. Ratiu. *Momentum maps and Hamiltonian reduction*, volume 222 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 2004.

[54] Juan-Pablo Ortega and Tudor S. Ratiu. Singular reduction of Poisson manifolds. *Lett. Math. Phys.*, 46(4):359–372, 1998.

[55] M. Perlmutter, M. Rodríguez-Olmos, and M. E. Sousa-Dias. On the geometry of reduced cotangent bundles at zero momentum. *J. Geom. Phys.*, 57(2):571–596, 2007.

[56] J. Pradines. Théorie de Lie pour les groupoïdes différentiables. Calcul différenetiel dans la catégorie des groupoïdes infinitésimaux. *C. R. Acad. Sci. Paris Sér. A-B*, 264:A245–A248, 1967.

[57] W. Sarlet, T. Mestdag, and E. Martínez. Lie algebroid structures on a class of affine bundles. *J. Math. Phys.*, 43(11):5654–5674, 2002.

[58] D. J. Saunders. *The geometry of jet bundles*, volume 142 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1989.

[59] J.-M. Souriau. *Structure of dynamical systems*, volume 149 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1997. A symplectic view of physics, Translated from the French by C. H. Cushman-de Vries, Translation edited and with a preface by R. H. Cushman and G. M. Tuynman.

[60] Paweł Urbański. An affine framework for analytical mechanics. In *Classical and quantum integrability (Warsaw, 2001)*, volume 59 of *Banach Center Publ.*, pages 257–279. Polish Acad. Sci., Warsaw, 2003.

[61] I. Vaisman. *Lectures on the geometry of Poisson manifolds*, volume 118 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1994.