GR and reduction

S. Capriotti

Palatini gravity and reduction

Dealing with variational problems in GR

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Talk contents

Where we are...

Topics included/not included

- **Included:**
  - Something about physics: GR and moving frames (vielbeins).
  - It will be defined variational problem for a field theory.
  - Geometry: Frame bundle and connection bundle.
  - "Generalized" reduction.
- **Missed:**
  - Physical motivations.
  - Exhaustive exploration of the existing literature.
  - Fun...

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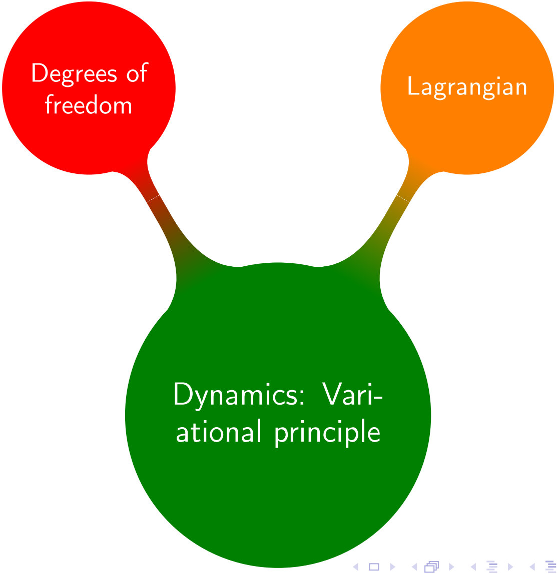
Elements of a field theory



Gravity as field theory

- **Einstein-Hilbert gravity:** Field = metric g,

$$\text{Lagrangian } = \int_M R[g]\sqrt{-g}d^4x,$$

$R[g]$ is the curvature for the Levi-Civita connection of g.

- **Palatini gravity:** Field = connection Γ in local terms,

$$\text{Lagrangian } = \int_M R[\Gamma]\wedge *(e\wedge e),$$

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Moving frames (vielbein)

- A moving frame is a collection of local vector fields, linearly independent and spanning TM on an open set.
- A connection is a pair $\left(e_k, \omega^i_j\right)$, where (e_j) is a moving frame and $\left(\omega^i_j\right)$ is a collection of 1-forms on M.
- Compatibility: Whenever $\left(\bar{e}_k, \bar{\omega}^i_j\right)$ is another pair and $\bar{e}_i = \sum_{j=1}^n g^j_i e_j$, then

$$\bar{\omega}^i_j = \sum_{k=1}^n h^i_k \mathbf{d} g^k_j + \sum_{k,l=1}^n h^i_k \omega^k_l g^l_j$$

donde $h = g^{-1}$.

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Palatini gravity

- Define $\eta := \text{diag}\,(-1,1,1,1)$, let (θ^i) be the base dual to (e_k). Local fields $x \mapsto \left(e_i, \omega^i_j\right)$, s.t. $\eta^{ip}\omega^j_p + \eta^{jp}\omega^i_p = 0, \mathbf{d}\theta^i + \omega^i_j \wedge \theta^j = 0.$
- Lagrangian

$$L[e,\omega] := \int_M \varepsilon_{ijkl}\eta^{iq}\theta^k \wedge \theta^l \wedge \left(\mathbf{d}\omega^j_q + \omega^j_p \wedge \omega^p_q\right).$$

- Dynamics: Variations are performed on e and the 1-forms ω independently.

Natural questions:

- Is L globally defined?
- Can the variations δe and $\delta\omega$ be considered independent?

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Field Theory [Gotay, 1991]

- A field is a section of a bundle Λ on M.
- An exterior differential system (EDS) is a differentially closed ideal in $\Omega^\bullet(\Lambda)$.
- A variational problem is a triple $(\Lambda \to M, \lambda, \mathscr{I})$.

Example:

First order classical field theory [Goldschmidt and Sternberg, 1973]:

$$\left(J^1E \to M, \mathscr{L}\mathbf{d}^n x, \mathscr{I}_{\text{con}} := \left\langle \mathbf{d}u^A - u^A_k \mathbf{d}x^k \right\rangle\right)$$

- Associated dynamical problem: Find the extremals of the functional

$$\sigma \mapsto \int_M \sigma^* \lambda$$

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The connection bundle [Castrillón López and Muñoz Masqué, 2001]

- Let $\tau : P \to M$ be a G-principal bundle.
- Connection as splitting

$$0 \longrightarrow \operatorname{ad} P \hookrightarrow TP/G \longrightarrow TM \longrightarrow 0, \qquad \sigma : TM \to TP/G$$

- Connection bundle $p : C(P) \to M$:

$$C(P)|_m := \{\rho : T_mM \to TP/G|_m \text{ tal que } T\tau \circ \rho = \mathrm{id}\}$$

- So $C(P) \equiv J^1P/G$.

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Canonical form on $p_G : J^1P \to C(P)$

- Given $\left[j^1_x s\right]_G \in C(P)$ we have the decomposition

$$T_u P = V_u P \oplus T_x s_0 \left(T_x M\right),$$

by choosing $s_0 : M \to P$ s.t.

$$s_0(x) = u$$
$$j^1_x s_0 \in \left[j^1_x s\right]_G.$$

Let $\Gamma_u : T_u P \to V_u P$ be the vertical projection.

- The map $\left[j^1_x s\right]_G \mapsto \Gamma$ is a bijection, and so

$$\left[j^1_x s\right]_G : T_u P \to V_u P \simeq \{u\} \times \mathfrak{g} \qquad u \in p_G^{-1}\left(\left[j^1_x s\right]_G\right).$$

- Canonical form on $J^1 P$ [García, 1972]

$$\omega|_{j^1_x s}(Y) := \left[j^1_x s\right]_G \left(T\tau_{10}(Y)\right).$$

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Properties of the canonical form [Castrillón López and Muñoz Masqué, 2001]

- Its components generate the contact structure on J^1P.
- Each connection $\sigma : M \to C(P)$ defines a unique map $\tilde{\sigma} : P \to J^1P$ s.t.
  - $\tilde{\sigma}$ is an equivariant map, and
  - it covers σ, namely

$$\begin{array}{ccc} P & \xrightarrow{\tilde{\sigma}} & J^1P \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle p_G} \\ M & \xrightarrow{\sigma} & C(P) \end{array}$$

The pullback $\tilde{\sigma}^*\omega \in \Omega^1(P, \mathfrak{g})$ is the connection form of σ.

- ω is a connection form on the bundle $J^1P \to C(P)$, with curvature

$$\Omega := \mathbf{d}\omega + \frac{1}{2}[\omega \overset{\wedge}{,} \omega]$$

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Palatini gravity Lagrangian in these terms

- We are looking for a n-form on

$$J^1LM = C(LM) \times_M LM,$$

possibly $SO(1, n-1)$-invariant.

- On $P = LM$ there exists another canonical form on J^1LM:

$$\theta := \tau_{10}^* \bar{\theta} \in \Omega^1\left(J^1LM, \mathbb{R}^n\right)$$

where $\bar{\theta}$ is the tautological 1-form of LM.

- It induces an additional universal form

$$\Theta := \mathbf{d}\theta + \omega \overset{\wedge}{\cdot} \theta \in \Omega^2\left(J^1LM, \mathbb{R}^n\right)$$

- Let us define [Dubois-Violette and Madore, 1987]

$$\theta_{i_1 \cdots i_p} := \frac{1}{(n-p)!} \varepsilon_{i_1 \cdots i_p i_{p+1} \cdots i_n} \theta^{i_{p+1}} \wedge \cdots \wedge \theta^{i_n}.$$

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$$\lambda_{PG} := \eta^{kp} \theta_{kl} \wedge \Omega^l_p, \qquad n \text{ arbitrary},$$

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Palatini gravity Lagrangian in these terms

- We are looking for a n-form on

$$J^1LM = C(LM) \times_M LM,$$

possibly $SO(1,n-1)$-invariant.

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First version

- $J^1LM \to M$
- $\lambda := \lambda_{PG}$
- $\mathscr{I}_1 := \left\langle \eta^{ip}\omega^j_p + \eta^{jp}\omega^i_p, \Theta^i \right\rangle$

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Some consecuences...

- E-L eqs ≡ vacuum Einstein eqs!
- The action of $H := SO(1, n-1)$ verifies

$$h \cdot \lambda_{PG} = \lambda_{PG}, \quad h \cdot \mathscr{I}_k \subset \mathscr{I}_k, k = 1, 2.$$

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EDS reduction [Anderson and Fels, 2005]

- Let $(\Lambda, \mathscr{I})$ be an EDS.
- Let G be a Lie group acting freely and properly on Λ.
- $g \cdot \mathscr{I} \subset \mathscr{I}$ for $g \in G$.
- On $\overline{\Lambda} := \Lambda/G$ can be defined

$$\overline{\mathscr{I}} := \left\{\alpha \in \Omega\left(\bar{\Lambda}\right) : p_G^*\alpha \in \mathscr{I}\right\}.$$

Reduced EDS

The reduced EDS of $(\Lambda, \mathscr{I})$ by the G-action is the EDS $\left(\overline{\Lambda}, \overline{\mathscr{I}}\right)$.

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Reduction of the contact structure on J^1P

- $P = M \times G$.
- $J^1P = G \times (T^*M \otimes \mathfrak{g})$ via $j^1_x s \mapsto \left(s(x), (T_x s)(s(x))^{-1}\right)$.
- In coordinates (g, ξ), the contact structure becomes

$$\mathscr{I}_{\text{con}} := \left\langle \mathbf{d}g \cdot g^{-1} - \xi, \frac{1}{2}[\xi \overset{\wedge}{,} \xi] - \mathbf{d}\xi \right\rangle_{\text{alg}}.$$

- Finally $J^1P/G = C(P) = T^*M \otimes \mathfrak{g}$.

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Reduction of a variational problem

Definition (Reduction for a variational problem $(\Lambda, \lambda, \mathscr{I})$)

Let G be a Lie group acting on Λ s.t.

- $\tau(g \cdot u) = \tau(u), u \in \Lambda, g \in G$,
- there exists $\overline{\lambda}$ s. t. $p_G^* \overline{\lambda} = \lambda$, and
- $g \cdot \mathscr{I} \subset \mathscr{I}$.

The reduced variational problem is $\left(\overline{\Lambda}, \overline{\lambda}, \overline{\mathscr{I}}\right)$, where

$$\overline{\Lambda} := \Lambda / G,$$

and $\overline{\mathscr{I}}$ is the reduced EDS for $\mathscr{I}$.

Euler-Poincaré reduction [Castrillón López et al., 2000]

- $L : J^1 P \to \mathbb{R}$ G-invariant, $\omega \in \Omega^n(M)$ volume form.
- Fix H connection on P.
- Let $l : C(P) \to \mathbb{R}$ be defined by the condition $p_G^* l = L$.
- For $s : M \to P$ and $\sigma := \left[j^1 s\right]_G$.
  Equivalent statements:
  - At s,
  $$\delta \int_M L\left(j_x^1 s\right) \omega = 0.$$
  - At σ,
  $$\delta \int_M l\left(\sigma\left(x\right)\right) \omega = 0.$$
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Relation with reduction of variational problems

Proposition

The reduced variational problem

$$\left(C(P), \overline{\mathscr{L}}\mathbf{d}^n x, \overline{\mathscr{I}_{\text{con}}} := \langle \Omega_2 \rangle \right).$$

is equivalent to the "weird" variational problem in Euler-Poincaré reduction.

Proof.

- $\sigma : M \to C(P)$ satisfies

$$\sigma^* \Omega_2 = 0 \qquad \Leftrightarrow \qquad \mathbf{d}_\sigma \sigma = 0 \qquad \Leftrightarrow \qquad \mathbf{d}_\sigma^2 = 0.$$

- Any variation is $\delta\sigma \in \Gamma\left(T^* M \otimes_M \operatorname{ad} P\right)$.
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Reduced Palatini gravity

- Reduce

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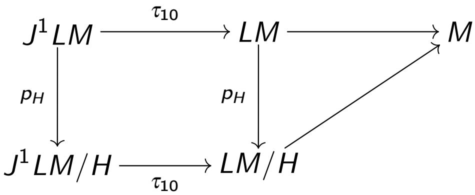

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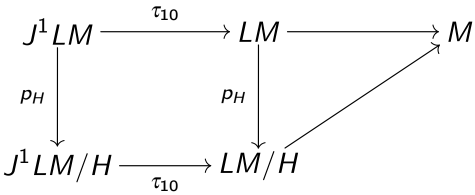

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Einstein-Hilbert gravity as reduced system

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$$\overline{\mathscr{I}_1} = \left\langle \mathbf{d}g^{\mu\nu} + \left(g^{\mu\sigma}\Gamma^{\nu}_{\gamma\sigma} + g^{\nu\sigma}\Gamma^{\mu}_{\gamma\sigma}\right)\mathbf{d}x^{\gamma}, \Gamma^{\sigma}_{\mu\nu} - \Gamma^{\sigma}_{\nu\mu}\right\rangle_{\text{dif}}.$$

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$$\overline{\lambda_{PG}} = \varepsilon_{\mu_1\cdots\mu_{n-2}\gamma\kappa}\sqrt{-\det g}\, g^{\kappa\phi}\mathbf{d}x^{\mu_1}\wedge\cdots\wedge\mathbf{d}x^{\mu_{n-2}}\wedge \\ \wedge\left(\mathbf{d}\Gamma^{\gamma}_{\rho\phi}\wedge\mathbf{d}x^{\rho} + \Gamma^{\sigma}_{\delta\phi}\Gamma^{\gamma}_{\beta\sigma}\mathbf{d}x^{\beta}\wedge\mathbf{d}x^{\delta}\right).$$

Proposition

$\left(J^1P/H, \overline{\lambda}_P, \overline{\mathscr{I}}_1\right)$ is an appropiate variational problem for Einstein-Hilbert gravity.

Proof: Reduced Palatini equivalent to Einstein-Hilbert

Proof.

- $J^1LM/H = C(LM) \times_M \Sigma$, $\Sigma \equiv$ bundle of metrics.
- The following commutes

$$\begin{array}{ccccc} J^1\Sigma & \xleftarrow{p_2} & C(LM)\times_M J^1\Sigma & \xrightarrow{\Pi} & C(LM)\times_M \Sigma \\ & \searrow & \downarrow & \swarrow & \\ & & M & & \end{array}$$

- Define $\mathscr{J} := \left\langle p_2^*\mathscr{I}^{\Sigma}_{\text{con}}, \Pi^*\overline{\mathscr{I}}_{PG}\right\rangle_{\text{diff}}$.
- There exists $L \subset C(LM) \times_M J^1\Sigma$ minimal w.r.t the property of containing every integral section of $\mathscr{J}$.
- $\left(J^1LM/H, \overline{\lambda}_{PG}, \overline{\mathscr{I}}_{PG}\right) \sim \left(L, \Pi^*\overline{\lambda}_{PG}\Big|_L, \mathscr{J}\big|_L\right) \sim \left(J^1\Sigma, \lambda_{EH}, \mathscr{I}^{\Sigma}_{\text{con}}\right)$, with $p_2^*\lambda_{EH} = \Pi^*\overline{\lambda}_{PG}\Big|_L$ $\square$.

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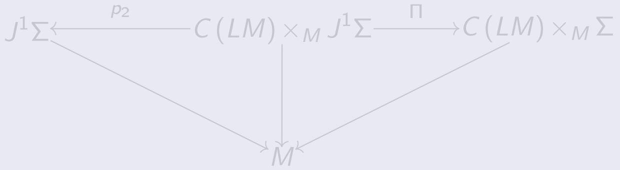

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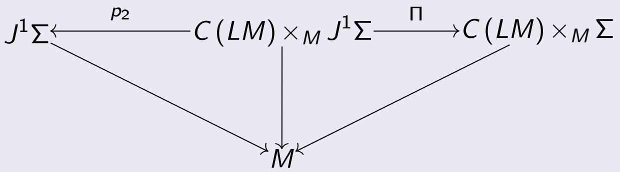

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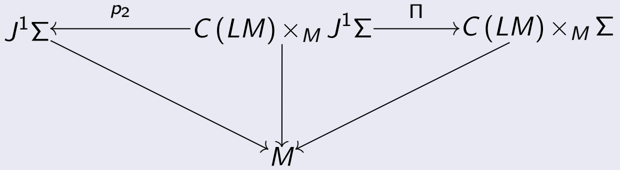

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Bonus track: A characterization of Levi-Civita connection

- Think about $\overline{\mathscr{I}_1}$ as a kind of contact structure on J^1LM/H.
- Every section $\sigma : M \to LM/H$ (a metric!) induces $\Sigma : M \to J^1LM/H$ s.t.
  - $\tau_{10} \circ \Sigma = \sigma$, (covers σ);

$$\begin{array}{ccc} & & J^1LM/H = C(LM) \times_M LM/H \\ & \overset{\Sigma}{\nearrow} & \downarrow \tau_{10} \\ M & \xrightarrow{\sigma} & LM/H \end{array}$$

  - $\Sigma^*\left(\overline{\mathscr{I}_1}\right) = 0$ (is an integral section of $\overline{\mathscr{I}_1}$).

$\Longrightarrow$ A characterization of the Levi-Civita connection!

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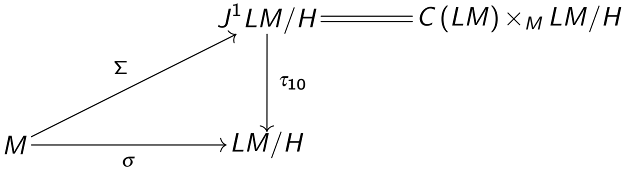

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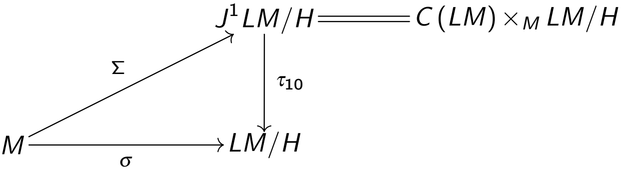

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