Abelian varieties in algebraic integrable systems

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- Algebraic integrable systems, Abelian varieties, and algebraic curves. Examples;
- The Painlevé property and the Kovalevskaya–Painlevé analysis for algebraic integrable systems. Strata of Jacobian varieties and related integrable systems;
- Lax representations, spectral curves, and linearization of integrable systems;
- The eigenvector map and the vector Baker functions;
- Symmetries of algebraic curves and Prym varieties, applications to integrable systems.

An example: The classical Euler top on $so(3)$

$$\dot{M} = M \times aM\,, \qquad M = (M_1, M_2, M_3)^T \in \mathbb{C}^3, \quad a = \mathrm{diag}(a_1, a_2, a_3)$$

Constants of motion

$$\langle M, aM\rangle = \underbrace{a_1 M_1^2 + a_2 M_2^2 + a_3 M_3^2 = I}_{\text{energy}}, \quad \langle M, M\rangle = \underbrace{M_1^2 + M_2^2 + M_3^2 = k^2}_{\text{momentum}}\,.$$

Complex invariant manifold $\mathcal{I}$ = intersection of 2 quadrics in $\mathbb{C}^3$
= an open subset of an elliptic curve

A (local) parametrization of $\mathcal{I}$ in terms of $\lambda \in \mathbb{C}$

$$M_i(\lambda) = k\sqrt{\frac{(a_j - c)(a_k - c)}{(a_i - a_j)(a_i - a_k)}}\sqrt{\frac{\lambda - a_i}{\lambda - c}}\,, \quad (i, j, k) = (1, 2, 3) \qquad c = I/k^2$$

In the real case, when $a_1 < a_2 < a_3$, one has $c \in [a_1, a_3]$.
The evolution of λ is given by the equation

$$\frac{d\lambda}{dt} = 2k\sqrt{(\lambda - a_1)(\lambda - a_2)(\lambda - a_3)(\lambda - c)}$$

hence λ is meromorphic on the elliptic curve
$E\ : \mu^2 = (\lambda - a_1)(\lambda - a_2)(\lambda - a_3)(\lambda - c)$

The Weierstrass form of E:

$$E \longleftrightarrow w^2 = 4(z-e_1)(z-e_2)(z-e_3), \quad e_1+e_2+e_3=0,$$

$$\frac{\lambda - a_i}{\lambda - c} = z - e_i, \quad i = 1,2,3.$$

$$u = \int_\infty^z \frac{dx}{2\sqrt{(x-e_1)(x-e_2)(x-e_3)}} \iff z = \wp(u|\omega_1,\omega_3),\ w = \frac{d}{du}\wp(u|\omega_1,\omega_3).$$

Here $\omega_1, \omega_3, \omega_2 = -\omega_1 - \omega_3$ are half-periods of $\wp$ and $e_i = \wp(\omega_i)$.

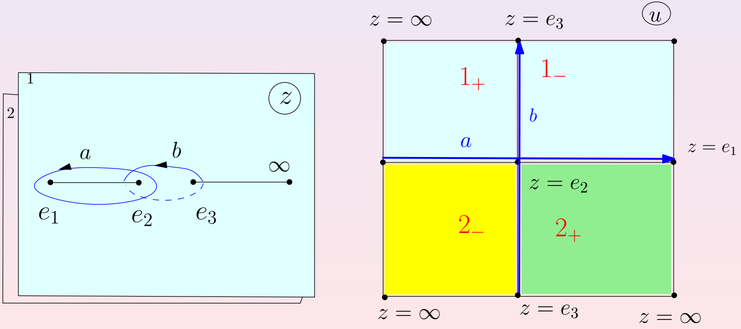


Figure: 2 faces of the elliptic curve

$M_i^2 = \alpha^2(z - e_i) = \alpha^2(\wp(u) - e_i)$ are meromorphic on E,
but each of $M_i = \alpha\sqrt{\wp(u) - e_i}$ is meromorphic on an unramified cover of E.
All M_1, M_2, M_3 are meromorphic on $\tilde{E}$: 4-fold cover of E obtained by doubling the periods $2\omega_1, 2\omega_3$. Its equation is the same as of E:

$$\tilde{E} = \left\{ W^2 = 4(Z - e_1)(Z - e_2)(Z - e_3) \right\},$$
$$Z = \wp(v \mid \Omega_1, \Omega_3), \; W = \wp'(v \mid \Omega_1, \Omega_3),$$

where $2\Omega_1 = 4\omega_1, 2\Omega_3 = 4\omega_3$ are the periods of $\tilde{E}$.
Rational parameterization of the momenta along $\mathcal{I}$ in terms of Z, W:

$$M_i = \beta_i \frac{Z^2 - 2e_i Z + e_i e_j + e_i e_k - e_j e_k}{W},$$
$$\beta_i = \frac{k\sqrt[6]{2(c - a_1)(c - a_2)(c - a_3)}}{\sqrt{(a_i - a_j)(a_i - a_k)}}, \qquad (i, j, k) = (1, 2, 3),$$

• Conclusion: the intersection of the two quadrics in $\mathbb{C}^3$ is $\tilde{E}$ without its 4 half periods $v = 0, \Omega_1, \Omega_2, \Omega_3$.

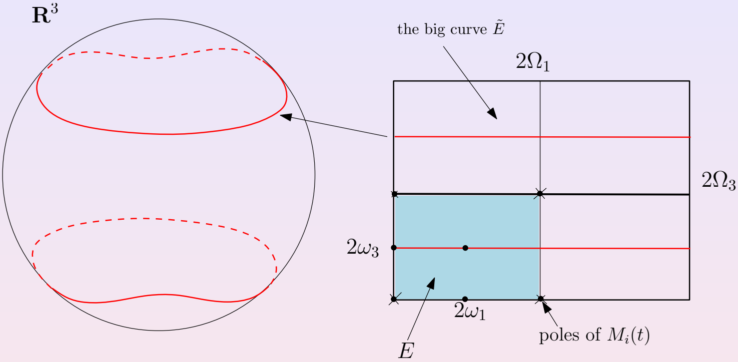


Figure: The parallelograms of periods of $E, \tilde{E}$ and the real trajectories on $\tilde{E}$ and in $\mathbb{R}^3$

Abelian varieties

Let $\Lambda = \{V_1\mathbb{Z} + \cdots + V_{2g}\mathbb{Z}\}$ be the lattice generated by $2g$ independent vectors $V_1, \ldots, V_{2g}$ in $\mathbb{C}^g(u_1, \ldots, u_g)$.

Def. The complex torus $\mathbf{A} = \mathbb{C}^g/\Lambda$ is *an Abelian variety* iff it can be algebraically embedded into $\mathbb{C}P^N(x_1 : \cdots : x_N : x_0)$:

$$\mathbf{A} = \{P_1(x) = 0, \ldots, P_k(x) = 0\}, \quad k \geq N - g$$

$\iff$ $\exists$ a basis $u_1, \ldots, u_g$ in $\mathbb{C}^g$ such that

$$\underbrace{(V_1 \ \cdots \ V_{2g})}_{\text{period matrix}} = \begin{pmatrix} \delta_1 & & & \\ & \ddots & & \mathcal{B} \\ & & \delta_g & \end{pmatrix}, \quad \mathcal{B}^T = \mathcal{B}, \quad \operatorname{Im}\mathcal{B} > 0,$$

$$\delta_i \in \mathbb{N}, \quad \underbrace{\delta_1 \geq \cdots \geq \delta_g}_{\text{polariztion}}$$

Examples. Elliptic curves, Jacobian varieties, Prym varieties.
Note: each Jacobian variety in principally polarized.

Jacobian varieties

Let S be a regular compact genus g curve, consider divisors

$$\mathcal{P} = n_1 P_1 + \cdots + n_k P_k, \quad n_s \in \mathbb{Z}, \quad P_s \in S, \quad \deg(\mathcal{P}) = n_1 + \cdots + n_k$$

$$\mathrm{Pic}^0(S) = \text{additive group of divisors of degree 0.}$$

Algebraic definition. $\mathrm{Jac}(S) = \mathrm{Pic}^0(S)/\{(f)\}$, where (f) is the divisor of zeros and poles of a meromorphic function f on S.

$$\mathcal{D}_1 \equiv \mathcal{D}_2 \iff \exists\, f(P) \text{ such that } (f) = \mathcal{D}_1 - \mathcal{D}_2.$$

$\dim \mathrm{Jac}(S) = \mathrm{genus}(S) = g$.

Analytic definition. Let $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ be a *canonical* basis of $H_1(S, \mathbb{Z})$ and $\underbrace{\{\omega_1, \ldots, \omega_g\}}_{\text{holomorphic differentials}}$ be a basis of $H^1(S, \mathbb{Z})$. Then

$\mathrm{Jac}(S) = \mathbb{C}^g(u_1, \ldots, u_g)/\Lambda$,

$$\Lambda = \{V_1\mathbb{Z} + \cdots + V_{2g}\mathbb{Z}\}, \quad V_1 = \oint_{a_1} \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_g \end{pmatrix}, \ldots, V_{2g} = \oint_{b_g} \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_g \end{pmatrix}$$

Relation between algebraic and analytic definition.

Algebraic Integrable Systems

$$\frac{d}{dt}Z = f(Z), \quad Z = (Z_1, \dots, Z_n) \in \mathbb{C}^n, \quad t \in \mathbb{C},$$

$$\mathcal{I}_h = \{\underbrace{H_1(X) = h_1, \dots, H_{n-g}(X) = h_{n-g}}_{\text{rational integrals}}\}$$

Following M.Adler, P. vanMoebeke, the system is ACI if

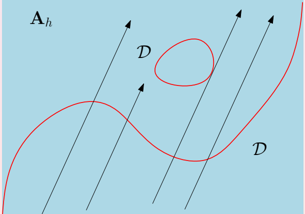
1) Its generic g-dimensional invariant manifolds $\mathcal{I}_h$ are open subsets of Abelian varieties

$$\mathcal{I}_h \subset \mathbf{A}_h, \quad \mathbf{A} = \mathbb{C}^g(u_1, \dots, u_g)/\Lambda, \quad \mathcal{I}_h = \mathbf{A}_h \setminus \mathcal{D}$$

2) The trajectories of the system are straight lines on $\mathbf{A}_h$: $u_i = c_i t + u_{i,0}$.

The variables Z_i are meromorphic functions of $u_1, \dots, u_g$ and of the complex t

Intersections of the trajectories with the Painlevé divisor D correspond to poles of the meromorphic solutions.



An example: The classical Neumann system on T^*S^2

C. Neumann. *De probleme quodam mechanico, quod ad primam integralium ultra-ellipticoram classem revocatum.* (1859).

The motion of a point on the unit sphere $S^2 = \{q_1^2 + q_2^2 + q_3^2 = 1\}$, under the action of the quadratic potential $U = a_1 q_1^2 + a_2 q_2^2 + a_3 q_3^2$:

$$\ddot{q} = aq - \nu q, \quad \nu = \langle q, aq \rangle + \langle \dot{q}, \dot{q} \rangle$$

Equivalently, on T^*S^2,

$$\dot{q} = p, \quad \dot{p} = aq - \nu q, \quad \nu = \langle q, aq \rangle + \langle p, p \rangle$$

Apart from $\langle q, q \rangle = 1$, $\langle q, p \rangle = 0$, there are integrals

$$F_i(q, p) = q_i^2 + \sum_{j \neq i}^{3} \frac{(p_i q_j - p_j q_i)^2}{a_j - a_i}, \quad i = 1, 2, 3,$$

$$F_1 + F_2 + F_3 = \langle q, q \rangle = 1.$$

Generic invariant manifolds are 2-dim. tori.

In the elliptic (spheroconical) coordinates λ_1, λ_2 on S^2 such that

$$q_i^2 = \frac{(a_i-\lambda_1)(a_i-\lambda_2)}{(a_i-a_j)(a_i-a_k)}, \quad \dot{q}_i = p_i = \frac{q_i}{2}\left(\frac{\dot\lambda_1}{a_i-\lambda_1}+\frac{\dot\lambda_2}{a_i-\lambda_2}\right), \quad (i,j,k)=(1,2,3)$$

the above integrals yield the quadratures

$$\frac{d\lambda_1}{2\sqrt{R(\lambda_1)}}+\frac{d\lambda_2}{2\sqrt{R(\lambda_2)}}=0,$$
$$\frac{\lambda_1 d\lambda_1}{2\sqrt{R(\lambda_1)}}+\frac{\lambda_2 d\lambda_2}{2\sqrt{R(\lambda_2)}}=dt,$$
$$R(\lambda)=-\lambda(\lambda-a_1)(\lambda-a_2)(\lambda-a_3)(\lambda-c_1)(\lambda-c_2)$$

Here the differentials $d\lambda/\sqrt{R(\lambda)}, \lambda d\lambda/\sqrt{R(\lambda)}$ can be regarded as holomorphic differentials on the genus 2 hyperelliptic curve $\Gamma = \{\mu^2 = R(\lambda)\}$.

The Abel map $\{P_1, P_2\} \to \mathrm{Jac}(\Gamma)$:
$$\begin{cases} \int_\infty^{P_1}\omega_1+\int_\infty^{P_2}\omega_1 = u_1 = u_{1,0}, \\ \int_\infty^{P_1}\omega_2+\int_\infty^{P_2}\omega_2 = u_2 = 2t+u_{2,0}, \end{cases}$$

$$\omega_1=\frac{d\lambda}{\sqrt{R(\lambda)}}, \quad \omega_2=\frac{\lambda d\lambda}{\sqrt{R(\lambda)}}, \quad P_j=(\lambda_j, \mu_j=\sqrt{R(\lambda_j)})\in\Gamma$$

Invariant varieties of the Neumann system

Symmetric rational functions of the coordinates $(\lambda_1, \mu_1), (\lambda_2, \mu_2)$ are meromorphic functions on $\mathrm{Jac}(\Gamma)$.

$$q_i^2 = \frac{(a_i - \lambda_1)(a_i - \lambda_2)}{(a_i - a_j)(a_i - a_k)}, \quad p_i = \frac{q_i}{\lambda_1 - \lambda_2}\left(\frac{a_i - \lambda_1}{\mu_1} - \frac{a_i - \lambda_2}{\mu_2}\right).$$

Thus $q_i^2, q_i p_i$, $i = 1, 2, 3$ are meromorphic on $\mathrm{Jac}(\Gamma)$, but q_i, p_i are not.

Theorem (D. Mumford)

The coordinates q_i, p_i are meromorphic on Abelian variety $\mathcal{I}$: an 8-fold unramified covering of $\mathrm{Jac}(\Gamma)$ obtained by doubling 3 of its 4 period vectors. The complex trajectories $q_i(t), p_i(t)$, $t \in \mathbb{C}$ are straight lines on $\mathcal{I}$.

The Kovalevskaya–Painlevé analisis

Weight-homogeneous system of differential equations

$$\frac{z_i}{dt} = f_i(z_1, \ldots, z_n), \quad 1 \le i \le n, \quad z_i, t \in \mathbb{C},$$

which are invariant with respect to similarity transformations

$$t \to t/\alpha, \quad z_1 \to \alpha^{g_1} z_1, \ \ldots, \ z_n \to \alpha^{g_n} z_n$$

with positive integer weights $g_1, \ldots, g_n$. Equivalently, the invariance condition are

$$f_i(\alpha^{g_1} z_1, \ldots, \alpha^{g_n} z_n) = \alpha^{g_i+1} f_i(z_1, \ldots, z_n)$$

Formal Laurent series solutions

$$z_i(t) = \frac{1}{t^{g_i}} \left(z_i^{(0)} + z_i^{(1)} t + \cdots \right), \qquad z_i^{(\alpha)} = \text{const}, \qquad i = 1, \ldots, n.$$

For the system to be *algebraically integrable*, the formal solutions must depend on $n-1$ free parameters !

Theorem (M.Adler, P. vanMoebeke, 1987)

If the weight-homogeneous polynomial system admits the formal solutions, then
1) the coefficients $z_i^{(m)}$ are determined from the equations

$$g_i z_i^{(0)} + f_i(z_i^{(0)}) = 0,$$

$$(K - mI)z^{(m)} = Y^{(m)}, \quad z^{(m)} = (z_1^{(m)}, \ldots, z_n^{(m)})^T, \qquad m = 1, 2, \ldots,$$

$$K_{ij} = \left(\frac{\partial f_i}{\partial z_j}(z^{(0)}) + \delta_{ij} g_i \right),$$

where $Y^{(m)}$ is a vector polynomial of $z^{(0)}, \ldots, z^{(m-1)}$,
2) the formal series are actually convergent.

- Coefficients $z_i^{(m)}$ are found by the recursive procedure. Following H. Yoshida, the matrix K and its eigenvalues $\rho_1, \ldots, \rho_n$ are called *the Kowalewski matrix* and *the Kowalewski exponents* respectively.
- The set $\mathcal{C} = \{z^{(0)}\}$ plays a key role in the Kowalewski method. Generally, it is not empty and may contain several connected components.
- There are as many different Laurent solutions as connected components of $\mathcal{C}$, and for each component the spectrum of the matrix K is fixed.

- If the Kowalewski matrix K is diagonalizable, each positive integer eigenvalue of K leads to a new free parameter in the Laurent series solution.
- -1 always belongs to the spectrum of K ($\rho_1 = -1$).

Conclusion: To obtain the full $(n-1)$-parameter family of meromorphic series solutions, K must be diagonalizable along $\mathcal{C}$ and the Kowalewski exponents $\rho_2, \ldots, \rho_n$ must be non-negative integers.

Theorem (H. Yoshida)

If $F(z)$ is a weight-homogeneous first integral of degree m, and $dF \neq 0$ on $\mathcal{C}$, then $\rho = m$ is a Kowalewski exponent.
Let the system possesses k weight-homogeneous integrals $F_1(z), \ldots, F_k(z)$ of the same degree m. Let $\beta_m \leq k$ be the rank of the Jacobi matrix $\|\partial F/\partial z\|$ at a point $c \in \mathcal{C}$.
Then $\rho = m$ is a Kowalewski exponent of multiplicity $\geq \beta_m$.

This result displays a remarkable relation between meromorphicity of a general solution of a weight-homogeneous system and the existence of its first integrals.

Warning: The integrals may be dependent on $\mathcal{C}$, then ρ_i may be different from their degrees.

A simple example: the classical Euler top again

$$\dot{M} = M \times aM, \qquad M = (M_1, M_2, M_3)^T, \quad a = \mathrm{diag}(a_1, a_2, a_3).$$

The equations are weight-homogeneous with $g_1 = g_2 = g_3 = 1$. Formal series solutions $M = \frac{1}{t}\left(M^{(0)} + M^{(1)}t + M^{(2)}t^2 + \cdots\right)$.
The leading term $M^{(0)} \in \mathbb{C}^3$ satisfies the system of three equations

$$M_\alpha + (a_\gamma - a_\beta)M_\beta M_\gamma = 0, \qquad (\alpha, \beta, \gamma) = (1, 2, 3).$$

having four point solutions

$$\mathcal{C}: \quad M^{(0)} = \left\{(\epsilon_1, \epsilon_2, \epsilon_3)^T,\ (-\epsilon_1, -\epsilon_2, \epsilon_3)^T,\ (-\epsilon_1, \epsilon_2, -\epsilon_3)^T,\ (\epsilon_1, -\epsilon_2, -\epsilon_3)^T\right\},$$

$$\epsilon_\alpha = i/\sqrt{(a_\alpha - a_\beta)(a_\alpha - a_\gamma)}.$$

Thus, the set $\mathcal{C}$ consists of four points. At each of them the quadratic energy and momentum integrals are independent.
Thus the Kowalewski exponents are -1,2,2.
The Laurent solutions are represented by the four series where $M^{(1)} = 0$, and $M^{(2)}$ depends on two free parameters. These 4 series correspond to expansions of the meromorphic solution of the Euler top near the 4 poles in the parallelogram of periods of $\tilde{E}$.

Integrable but non-ACI systems

- Inversion of a single hyperelliptic integral $t = \int_\infty^x \frac{d\lambda}{\sqrt{R(\lambda)}}$, deg $R(\lambda) > 4$
- The Neumann system on $S^2 = \{q_1^2 + q_2^2 + q_3^2 = 1\}$ with a higher degree *separable* potential:

$$H(p,q) = \frac{1}{2}\langle p, p\rangle + \varkappa \mathcal{U}_n(q), \quad \mathcal{U}_2 = \frac{1}{2}\langle aq, q\rangle,$$

$$\mathcal{U}_4 = \langle q, aq\rangle^2 - 2\mathrm{Tr}\, a\langle q, aq\rangle - \langle q, a^* q\rangle, \quad a^* = \mathrm{diag}(a_2 a_3, a_1 a_3, a_1 a_2).$$

In the spheroconical coordinates λ_1, λ_2 on S^2 such that

$$q_i^2 = \frac{(a_i - \lambda_1)(a_i - \lambda_2)}{(a_i - a_j)(a_i - a_k)}, \quad \dot{q}_i = p_i = \frac{q_i}{2}\left(\frac{\dot\lambda_1}{a_i - \lambda_1} + \frac{\dot\lambda_2}{a_i - \lambda_2}\right)$$

the equations of motion reduce to the quadratures

$$\frac{d\lambda_1}{2\sqrt{\mathcal{R}(\lambda_1)}} + \frac{d\lambda_2}{2\sqrt{\mathcal{R}(\lambda_2)}} = 0,$$
$$\frac{\lambda_1 d\lambda_1}{2\sqrt{\mathcal{R}(\lambda_1)}} + \frac{\lambda_2 d\lambda_2}{2\sqrt{\mathcal{R}(\lambda_2)}} = dt,$$

$$\mathcal{R}(\lambda) = -\lambda(\lambda - a_1)(\lambda - a_2)(\lambda - a_3)[(\lambda - c_1)(\lambda - c_2) - \varkappa\lambda^{(n-2)/2}].$$

They involve 2 holomorphic differentials of the curve $\mu^2 = \mathcal{R}(\lambda)$ of genus > 2 ! The quadratures cannot be inverted in terms of meromorphic functions of t.

Strata of Jacobians

Given a genus 2 curve $G = \{F(\lambda, \mu) = 0\}$ with holomorphic differentials $\omega_1, \dots, \omega_g$ and k points $P_j = (\lambda_j, \mu_j)$, $j = 1, \dots, k < g$.

Def. k-stratum W_k in $\mathrm{Jac}(G)$: the image of the *incomplete* Abel map

$$\mathcal{A}(P_1, \dots, P_n | p_0) \ : \ \int_{p_0}^{P_1} \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_g \end{pmatrix} + \cdots + \int_{p_0}^{P_n} \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_g \end{pmatrix} = \begin{pmatrix} u_1 \\ \vdots \\ u_g \end{pmatrix} \in \mathrm{Jac}(G).$$

In particular, the stratum W_{g-1} is a translation of the *theta-divisor* $\Theta = \{\theta(u_1, \dots, u_g) = 0\} \subset \mathrm{Jac}(G)$.

Theorem

The generic complex invariant manifolds of the Neumann system on S^2 with the potential $\varkappa \mathcal{U}_n(q)$ is an open subset of W_2 of the $(2 + (n-2)/2)$-dimensional Jacobian of $\Gamma = \{\mu^2 = \mathcal{R}(\lambda)\}$. Its complex solutions have movable singularities $t^{1/d}$, d depends on the degree n.

Lax representation and spectral curves

Given a system of differential equations

$$\frac{dx_i}{dt} = f_i(x_1, \dots, x_n), \quad 1 \le i \le n, \quad x_i, t \in \mathbb{C}, \qquad (*)$$

a matrix equation $\dot{L}(x,\lambda) = [L(x,\lambda), N(x,\lambda)]$ with a parameter $\lambda \in \mathbb{C}$, is a Lax pair for the system (*) if the latter converts the equation to identity.

Note: the Lax pair may or may not imply the system (*) uniquely.

Let *the Lax matrix* $L(x,\lambda)$ depend on λ in a rational way. Its spectral curve S is given by the characteristic equation

$$F(\lambda,\mu|x) = |L(x,\lambda) - \mu\mathbf{I}| = 0, \quad S \ : \ \{F(\lambda,\mu) = 0\} \subset \mathbb{P}^2.$$

The coefficients of S provide first integrals for the system (*).

Note: the same system (*) may have different Lax pairs (e.g., of different dimension) and different spectral curves, which may be or may be not birationally equivalent.

2×2 matrix Lax pair for the Neumann system on S^{n-1}

$$\dot q = p, \quad \dot p = Aq + \nu q, \quad \nu = -\langle p, p\rangle - \langle q, Aq\rangle, \qquad A = \mathrm{diag}(a_1, \dots, a_n),$$

$$\dot L(\lambda) = [L(\lambda), N(\lambda)], \qquad L(\lambda) = \begin{pmatrix} \sum_{i=1}^n \frac{q_i p_i}{\lambda - a_i} & \sum_{i=1}^n \frac{q_i^2}{\lambda - a_i} \\ 1 - \sum_{i=1}^n \frac{p_i^2}{\lambda - a_i} & -\sum_{i=1}^n \frac{q_i p_i}{\lambda - a_i} \end{pmatrix},$$

$$N(\lambda) = \begin{pmatrix} 0 & 1 \\ -\lambda + \nu(p, q) & 0 \end{pmatrix}.$$

Let $a(\lambda) = (\lambda - a_1) \cdots (\lambda - a_n)$. For the polynomial Lax matrix $\mathbf{L}(\lambda) = a(\lambda) L(\lambda)$ the characteristic equation $|\mathbf{L}(\lambda) - \mu \mathbf{I}_2| = 0$ has the form

$$\mu^2 = -a^2(\lambda) \left(\sum_{i<j}^n \frac{(q_i p_j - q_j p_i)^2}{(\lambda - a_i)(\lambda - a_j)} - \sum_{i=1}^n \frac{q_j^2}{\lambda - a_i} \right),$$

and gives the odd order hyperelliptic curve Γ of genus $g = n - 1$

$$\mu^2 = -a(\lambda)\,(\lambda - c_1) \cdots (\lambda - c_{n-1}),$$

$c_1, \dots c_{n-1}$ being constants of motion.
An $n \times n$ Lax pair, giving an equivalent spectral curve was found by Y. Moser.

The eigenvector map

Let $\mathbb{L}(x(t),\lambda)$ be a $d\times d$ polynomial Lax matrix and $S=\{|\mathbb{L}(x,\lambda)-\mu\mathbf{I}|=0\}$ be its spectral curve (with its infinite points), assumed to be regular or regularized, of geometric genus g.

$$\lambda\in\mathbb{C}\longrightarrow\{(\lambda,\mu_1),\ldots(\lambda,\mu_d)\}\in S$$

$$\mathbb{L}(\lambda)\psi_k=\mu_k\psi_k,\quad k=1,\ldots,d\implies\psi(\lambda,\mu_k)\longrightarrow\psi(P),\ P\in S$$

Thus we have the eigenvector bundle $\varepsilon:S\longrightarrow\mathbb{P}^{d-1}$ given by the eigenvectors $\psi(P)=(\psi^1(P),\ldots,\psi^d(P))^T$.
Impose *normalization* $\langle\alpha,\psi(P)\rangle=1$, $\alpha\in\mathbb{P}^{d-1}$ is an arbitrary constant, and consider the *minimal* effective divisor $\mathcal{D}_\alpha$ on S such that

$$(\psi^l(P))=\text{zeros of }\psi^l(P)\text{ - poles of }\psi^l(P)\geq-\mathcal{D}_\alpha,\quad l=1\ldots,d,$$

that is, each of $\psi^l(P)$ can have poles *at most* at the points of $\mathcal{D}_\alpha$.
Note: $\deg(\mathcal{D}_\alpha)=g+d-1=N$.
(One can always chose such a normalization α that $d-1$ points of $\mathcal{D}_\alpha$ will be fixed in the infinite part of S.)

Algebraic calculation of divisor $\mathcal{D}_\alpha$ on S

Solve the system of d algebraic equations $\underbrace{(\mathbb{L}(\lambda) - \mu\mathbf{I})^*}_{\text{cofactor matrix}} \alpha = 0$

Example: 2×2 Lax matrix for the Neumann system on S^{n-1}

$$\mathbf{L}(\lambda) = a(\lambda)L(\lambda) = \begin{pmatrix} V(\lambda) & U(\lambda) \\ W(\lambda) & -V(\lambda) \end{pmatrix},$$

$$U(\lambda) = a(\lambda)\sum_{i=1}^{n} \frac{q_i^2}{\lambda - a_i} = (\lambda - \lambda_1)\cdots(\lambda - \lambda_g) = \lambda^g + u_1\lambda^{g-1} + \cdots + u_g,$$

$$V(\lambda) = v_1\lambda^{g-1} + \cdots + v_g, \qquad g = n-1,$$

$$W(\lambda) = \lambda^{g+1} + w_0\lambda^g + w_1\lambda^{g-1} + \cdots + w_g = (\lambda - \nu_1)\cdots(\lambda - \nu_{g+1}),$$

Note: $\lambda_1, \ldots, \lambda_{n-1}$ are spheroconical coordinates on $S^{n-1} = \langle q, q \rangle$.
Let $\alpha = (1,0)^T$. Then

$$(\mathbf{L}(\lambda) - \mu\mathbf{I})^*\alpha = 0 \implies \begin{pmatrix} -V(\lambda) - \mu & -W(\lambda) \\ -U(\lambda) & V(\lambda) - \mu \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

and $\mathcal{D}_\alpha = \{U(\lambda) = 0, V(\lambda) = -\mu\} = \{(\lambda_1, \mu_1) \ldots, (\lambda_{n-1}, \mu_{n-1}), \infty\}$.

The eigenvector map

Divisors $\mathcal{D}_\alpha$, $\mathcal{D}_{\alpha'}$ corresponding to different normalizations $\alpha, \alpha' \in \mathbb{P}^{d-1}$ are linearly equivalent:
$\exists$ a meromorphic function $F(P)$ on S such that $(F) = \mathcal{D}_\alpha - \mathcal{D}_{\alpha'}$.
Therefore, for a basepoint $p_0 \in S$, the degree zero divisors $\mathcal{D}_\alpha - Np_0, \mathcal{D}_{\alpha'} - Np_0$ give the same point in $\mathrm{Jac}(S)$, more precisely, in the open subset $\mathrm{Jac}(S) \setminus \Theta$, with $\Theta \in \mathrm{Jac}(S)$ being a translate of the theta-divisor, the Abel image of all the special divisors on S.

Let now $\mathcal{I}_S$ be the isospectral manifold: the set of all the above matrices $\mathbb{L}(\lambda)$ having the same structure and the same spectral curve S. Thus we get *the eigenvector map*

$$\mathcal{E} : \mathcal{I}_S \to \mathrm{Jac}(S) : \quad \mathbb{L}(\lambda) \to \psi(P) \to \{\mathcal{D}\} \to (\mathcal{D}_\alpha - Np_0) \in \mathrm{Jac}(S)$$

Reconstruction of $\mathbb{L}(\lambda)$: Given $\{\mathcal{D}\} = \mathcal{E}(\mathbb{L}(\lambda))$, the matrix $\mathbb{L}$ can be reconstructed up to a conjugation by an element of $\mathbb{PGL}(d, \mathbb{C})$ (not depending of λ), which preserves the structure of $\mathbb{L}(\lambda)$:

$$\mathcal{D} \longrightarrow \underbrace{L(\mathcal{D}) = \{f(P) \mid (f) \geq \mathcal{D}\}}_{\dim = N - g + 1 = d} = \{f_1 = 1, f_2(P), \dots, f_d(P)\}$$

$$\longrightarrow \vec{\mathbf{f}}(P) = (f_1, \dots, f_d)^T \longrightarrow \Psi(\lambda) = (\mathbf{f}(P_1) \cdots \mathbf{f}(P_d)), \quad P_j = (\lambda, \mu_j)$$

$$\longrightarrow \mathcal{X}(\lambda) = \Psi(\lambda)\, \mathrm{diag}(\mu_1, \dots, \mu_d)\, \Psi^{-1}(\lambda)$$

Theorem (B. Dubrovin, I. Krichiver)

$\mathcal{X}(\lambda) = \Psi(\lambda)\,\mathrm{diag}(\mu_1,\ldots,\mu_d)\,\Psi^{-1}(\lambda)$ *is independent of the order of* $\mu_1,\ldots,\mu_d$*, is, in fact, polynomial in* λ*, and has the prescribed spectral curve* S*. It reconstructs the matrix* $\mathbb{L}(\lambda)$ *up to a conjugation by a constant matrix.*
The induced map

$$\mathcal{M} \ : \ \{\mathbb{L}(\lambda) \in \mathcal{I}_S \text{ up to conjugation by a matrix of } \mathbb{PGL}(d,\mathbb{C})\} \mapsto \mathrm{Jac}(S)\backslash\Theta$$

is injective.

Important example: $\mathbb{L}(x,\lambda) = J\lambda^n + A_{n-1}(x)\lambda^{n-1} + \cdots + A_{n-1}(x)$, $J = \mathrm{diag}(J_1,\ldots,J_d)$ =const and *there are no other restrictions on* $A_{n-1}(x),\ldots,A_{n-1}(x)$.

$\mathbb{L}(\lambda) \to \mathcal{G}\mathbb{L}(\lambda)\mathcal{G}^{-1}$ does not change the structure of $\mathbb{L}(\lambda)$ iff $\mathcal{G} \in \underbrace{\mathbb{C}^* \times \cdots \times \mathbb{C}^*}_{d \text{ copies}}$

$\Longrightarrow$ Isomorphism $\mathcal{I}_S/(\mathbb{C}^*)^d \longleftrightarrow \mathrm{Jac}(S) \setminus \Theta$

Linearization on Jac(S) (when possible)

From the Lax pair

$$\frac{d}{dt}\mathbb{L}(x,\lambda) = [\mathbb{L}(x,\lambda), \mathcal{N}(x,\lambda)] \iff [\frac{d}{dt} + \mathcal{N}(x,\lambda), \mathbb{L}(x,\lambda)] = 0$$

it follows $\frac{d}{dt}\psi(P,t) + \mathcal{N}(x,\lambda)\psi(P,t) = f(P,t)\psi(P,t)$, $P \in S$.
Use normalization

$$\langle \alpha, \psi(P,t)\rangle \equiv 1 \implies \langle \alpha, \dot{\psi}(P,t)\rangle = 0 \implies f(P,t) = \langle \alpha, \mathcal{N}\psi(P,t)\rangle$$

Let $\vec{\omega} = (\omega_1, \ldots, \omega_g)^T$ be a basis of holomorphic differentials on the spectral curve S having infinite points $\infty_1, \ldots, \infty_k$, $k \le d$.
Let the divisor $\mathcal{D}_t$ on S be the image of the eigenvector map for $\mathbb{L}(x(t), \lambda)$.

Theorem (Linearization Criterion (Adler, van Moerbeke, Vanhaecke))

$A\ (\ D_t) = \int_{\mathcal{D}_0}^{\mathcal{D}_t} \vec{\omega} = t \sum_{i=1}^{k} \mathrm{Res}_{\infty_i}(f(P,t)\,\omega_1, \ldots, f(P,t)\,\omega_g)^T = \mathbf{U}t.$

Note: the function f may depend on α and on t, but the flow of the system is linearized on Jac(S) iff the above residua do not depend on t.

A relevant problem

Given a genus g algebraic curve S and a *generic* effective divisor $\mathcal{D}$ of degree N on it, according to the Riemann-Roch theorem,
the space $L(\mathcal{D})$ of meromorhic functions which may have poles at most at $\mathcal{D}$ has dimension $N - g + 1$.

Exersize 1. Given an elliptic curve $E\ :\ \{\mu^2 = (\lambda - a_1)(\lambda - a_2)(\lambda - a_3)\}$, and a single point $Q \in E$, we have $L(Q) = \{1\}$. Now, given 2 *generic* points $Q_1 = (\lambda_1, \mu_1), Q_2 = (\lambda_2, \mu_2)$ on E (they do not coinside with the branch points $(a_i, 0)$), to find explicitly the space $L(Q_1, Q_2)$.

Solution: $L(Q_1, Q_2) = \{1, f(\lambda, \mu)\}$ with

$$f = \frac{\mu + V(\lambda)}{(\lambda - \lambda_1)(\lambda - \lambda_2)}, \quad V(\lambda) = \mu_1 \frac{\lambda - \lambda_2}{\lambda_1 - \lambda_2} + \mu_2 \frac{\lambda - \lambda_1}{\lambda_2 - \lambda_1}.$$

Indeed, the denominator has simple zeros at Q_1, Q_2 as well as at $\bar{Q}_1 = (\lambda_1, -\mu_1), \bar{Q}_2 = (\lambda_2, -\mu_2)$, whereas the numerator vanishes at 3 points $P, \bar{Q}_1, \bar{Q}_2$ and not at Q_1, Q_2. Hence f has poles at Q_1, Q_2 only.
Note that at $\infty \in E$ the numerator has pole of degree 3, whereas the denominator has a pole of degree 4. As a result,
$(f) = \infty + P - Q_1 - Q_2$. (We do not give explicitly P here.)

Reconstruction of $\mathbb{L}(x, \lambda)$ via the Baker functions

Suppose that the integrable system admits linearization on Jac(S).

Given the Lax pair $[\frac{d}{dt} + \mathcal{N}(x, \lambda), \mathbb{L}(x, \lambda)] = 0$, we want to find the eigenvector function $\psi(P, t)$, $P = (\lambda, \mu) \in S$ satisfying

$$\frac{d}{dt}\psi(P, t) = -\mathcal{N}(x(t), \lambda)\,\psi(P, t), \quad \mathbb{L}(x(t), \lambda)\,\psi(\underbrace{(\lambda, \mu_i)}_{P_i}, t) = \mu_i\,\psi((\lambda, \mu_i), t),$$

$$i = 1, \dots, d$$

$$\text{Then} \quad \Psi(\lambda, t) = (\psi(P_1, t) \cdots \psi(P_d, t)), \quad P_j = (\lambda, \mu_j)$$

$$\longrightarrow \mathcal{X}(\lambda, t) = \Psi(\lambda, t)\,\mathrm{diag}(\mu_1, \dots, \mu_d)\,\Psi^{-1}(\lambda, t)$$

and $\mathcal{X}(\lambda, t)$ recovers $\mathbb{L}(x(t), \lambda)$ up to conjugation by a *constant* matrix.

Recall: $(\psi_i(P, 0)) \geq -\mathcal{D}_0$, $(\psi_i(P, t)) \geq -\mathcal{D}_t$.

Assume that $\mathcal{N}(x, \lambda)$ is a matrix polynomial in λ with a constant leading matrix coefficient. Then the components of $\psi(P, t)$ must have essential singularities at (some of) the infinite points of S.

Reconstruction of $\mathbb{L}(x, \lambda)$ via the Baker functions II

Let S have k infinite points $\infty_1, \ldots, \infty_k$.
Let $a_1, b_1, \ldots, a_g, b_g$ be a canonical basis of cycles on S.

A scalar k-point Baker–Akhieser function on S with poles at $\{P_1, \ldots, P_g\} \subset \mathcal{D}_0$ and essential singularities at $\infty_1, \ldots, \infty_k$:

$$\phi(P,t) = C(t) \cdot \exp\left(t \int_{p_0}^{P} \Omega_\infty \right) \frac{\theta(\mathcal{A}(P) - \mathcal{A}(P_1, \ldots P_g) + \mathbf{U}t - \mathcal{K})}{\theta(\mathcal{A}(P) - \mathcal{A}(P_1, \ldots P_g) - \mathcal{K})},$$

Ω_∞ is a normalized meromorhic diff. with poles at $\infty_1, \ldots, \infty_k$ such that

$$\mathbf{U} = \left(\oint_{b_1} \Omega_\infty, \ldots, \oint_{b_g} \Omega_\infty \right)^T \longleftarrow \text{vector of } b\text{-periods of } \Omega_\infty,$$

$$\mathbf{U}t = \int_{\mathcal{D}_0}^{\mathcal{D}_t} \vec{\omega}, \quad \mathcal{A}(P) = \int_{p_0}^{P} \vec{\omega}, \quad \mathcal{K} \text{ is the vector of Riemann constants}$$

Note: due to quasi-periodic properties of θ, when the path from p_0 to P changes by a cycle on S, $\phi(P,t)$ does not change. So, it is single-valued on $S \setminus \{\infty_1, \ldots, \infty_k\}$.

vector Baker function

Now we want to have d-dim. vector $\psi(P,t)$ whose components

$$\psi^1(P,t),\ \ldots,\ \psi^{d-1}(P,t),\ \psi^d(P,t),$$

can have poles at most at $\mathcal{D}_0 = \{P_1, \ldots, P_g, \ldots, P_N\}$
(recall: $\deg(\mathcal{D}_0) = g + d - 1 = N$.)
Then one can shoose

$$\psi^1(P,t) = \frac{C_1(t)}{C(t)}\phi(P,t),$$

$$\psi^2(P,t) = \frac{C_2(t)}{C(t)}\phi(P,t) \text{ with } P_1, \ldots, P_g \text{ replaced by } P_1, \ldots, P_{g-1}, P_{g+1},$$

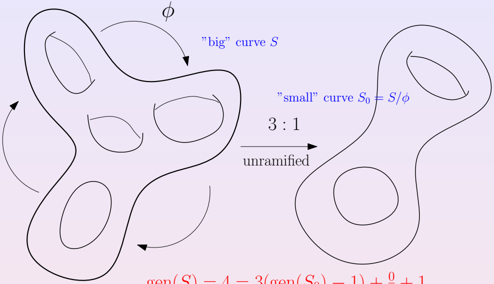
$$\vdots$$

$$\psi^d(P,t) = \frac{C_d(t)}{C(t)}\phi(P,t) \text{ with } P_1, \ldots, P_g \text{ replaced by } P_1, \ldots, P_{g-1}, P_{g+d-1}$$

The coefficients $C_1(t), \ldots C_d(t)$ are found from the above normalization $\langle \alpha, \psi(P,t) \rangle \equiv 1$.

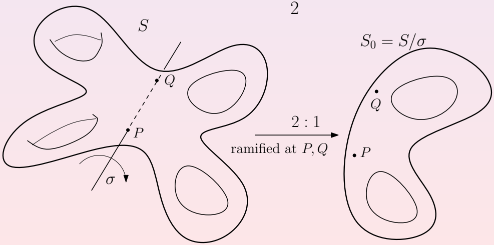
Prym Varieties

Examples of automorphisms of algebraic curves S and coverings $S \to S_0$



$\text{gen}(S) = 4 = 3(\text{gen}(S_0) - 1) + \frac{0}{2} + 1$

2



$\text{gen}(S) = 4 = 2(\text{gen}(S_0) - 1) + \frac{2}{1} + 1$

2

Prym Varieties

Let $\phi : S \to S$ be an automorphism of the curve S of genus g and $S_0 = S/\phi$ be the factor curve of genus g_0. Then $\text{Jac}(S)$ contains 2 Abelian subvarieties of complimentary dimensions

$$\text{Jac}(S_0) \subset \text{Jac}(S) \supset \underbrace{\text{Prym}(S, \phi)}_{\text{dim}=g-g_0}$$

The automorphism ψ extends to $\text{Jac}(S)$, then $\text{Jac}(S_0)$ is invariant with respect to ϕ.

If ϕ is an involution σ, then $\text{Prym}(S, \phi)$ is *anti-invariant* with respect to σ. Note that $\text{Jac}(S)$ is NOT isomorphic to the direct product $\text{Jac}(S_0) \otimes \text{Prym}(S, \phi)$ but only *isogeneous* to it.

In the general case of coverings $S \to S_0$ (with more than 2 branch points), the subvariety $\text{Prym}(S, \phi)$ is not the Jacobian variety of an alg. curve just because it does not have a principal polarization.

On the other hand, Prym varieties often appear as complex invariant manifolds of algebraic integrable systems, and it may be nesesary to have an algebraic description of $\text{Prym}(S, \phi)$, relating it to a Jacobian variety via an isogeny.

General case of 2-fold covering ramified at 2 points

Consider hypereliptic genus g curve $C : y^2 = f(x)$, *f has simple roots and an odd degree.*

Theorem *Any 2-fold covering $S \to C$ ramified at 2 finite points $P = (x_P, y_P), Q = (x_Q, y_Q) \in C$ can be written*

$$S : z^2 = y + h(x), \quad y^2 = f(x), \quad \text{involution } \sigma : (x, y, z) \to (x, y, -z),$$

where $h(x)$ is a polynomial of degree $\leq g + 1$ such that

$$h^2(x) - f(x) = (x - x_P)(x - x_Q)\rho^2(x)$$

for some polynomial $\rho(x)$. Then gen$(S) = 2g$ and

1) $2g$-dim. Jacobian of S contains two g-dimensional Abelian subvarieties: Jac(C) and Prym(S, σ),

2) Prym(S, σ) is *principally polarized* and is the Jacobian of a hyperelliptic curve C' (D.Mumford, S. Dalaljan, 1975),
3) (A. Levin, 2012) C' can be written in form

$$w^2 = h(x) + Z, \quad Z^2 = h^2(x) - f(x) \equiv (x - x_P)(x - x_Q)\rho^2(x),$$

which is equivalent to the plane curve $w^4 - 2h(x)w^2 + f(x) = 0$.
(The latter can be transformed to a hyperelliptic form.)

General trigonal genus 4 curve S over genus 2 curve C

Genus 4 trigonal curve S in $\mathbb{C}^2 = (z, W)$ with $\sigma : (z, \mu) \to (-z, -\mu)$

$$\mu^3 + az\mu^2 + b\mu z^2 + c\mu + dz^5 + ez^3 + kz = 0,$$
$$a, b, c, d, e, k = \text{const} \in \mathbb{C}$$

Introducing the new variable $x = z/\mu$, which is invariant with respect to σ, the equation of S rereads

$$dz^4x^3 + (1 + ax + bx^2 + ex^3)z^2 + kx^3 + cx^2 = 0.$$

Solving it with respect to z^2, we get the equation

$$z^2 = \frac{-h(x) + \sqrt{f(x)}}{2\,d\,x}, \quad h(x) = ex^3 + bx^2 + ax + 1,$$
$$f(x) = \alpha x^6 + \beta x^5 + \gamma x^4 + \delta x^3 + \varepsilon x^2 + 2ax + 1,$$
$$\alpha = e^2 - 4dk,\ \beta = 2be - 4dc,\ \gamma = b^2 + 2ea,\ \delta = 2e + 2ab,\ \varepsilon = a^2 + 2b.$$

Thus S is 2-fold covering of the genus 2 curve C: $\{y^2 = f(x)\}$ ramified at $P = (0, 0), Q = (\infty, \infty) \in S$ that are projected to the points on C

$$P = \left(x_P = -\frac{c}{k}, \quad y_P = 1 - \frac{ca}{k} + \frac{c^2b}{k^2} - \frac{c^3e}{k^3}\right), \quad Q = (x_Q = 0, \quad y_Q = 1).$$

Theorem

1) If $f(x)$ has an even degree $2g+2$ and $g = 2, 4, 6, \dots$, any covering $\widetilde{C} \to C : \{y^2 = f(x)\}$ ramified at $P = (x_P, y_P), Q = (x_Q, y_Q) \in C$ can be written in the form

$$\left\{ y^2 = f(x), \quad z^2 = \frac{y + h(x)}{x - x_P} \quad \text{or, equivalently,} \quad z^2 = \frac{y + h(x)}{x - x_Q} \right\},$$

where $h^2(x) - f(x) = (x - x_P)(x - x_Q)\rho^2(x)$ for some polynomial $\rho(x)$.

2) Prym$(\widetilde{C}, \sigma)$ is isomorphic to the Jacobian of a genus g hyperelliptic curve C'

$$\Big\{ Y^2 = h^2(x) - f(x) = (x - x_P)(x - x_Q)\rho^2(x),$$
$$w^2 = \frac{Y + h(x)}{x - x_P} \quad \text{or, equivalently,} \quad w^2 = \frac{Y + h(x)}{x - x_Q} \Big\}.$$

The latter is transformed to the form $v^2 = P_{2g+2}(u)$, by the birational transformation

$$x = \frac{x_Q u^2 - x_P}{u^2 - 1}, \quad w = \left(\frac{x_Q - x_P}{u^2 - 1} \right)^{g/2} v,$$

Prym variety in the case of 4 branch points

Given genus 3 curve C

$$y^2 = h_2 x^2 + h_1 x + h_0 + 2h_3 \sqrt{(x-c_1)(x-c_2)(x-c_3)}\}$$

which is 2:1 covering of the elliptic curve
$E : \{w^2 = (r-c_1)(r-c_2)(r-c_3)\}$ under the involution
$\sigma : (x,y) \to (x,-y)$.
Covering $C \to E$ ramified at 4 points $Q_1, \ldots, Q_4$ on E.

Jac(C) contains the 2-dim. Prym variety Prym(C,σ) with polatization (1,2), its period matrix

$$\Lambda = \begin{bmatrix} 1 & 0 & a & b \\ 0 & 2 & b & c \end{bmatrix}, \quad \text{equivalent to} \quad \widehat{\Lambda} = \begin{bmatrix} 2 & 0 & c & b \\ 0 & 1 & b & a \end{bmatrix},$$

Dual Prym variety Prym$^*(C,\sigma)$

$$\widehat{\Lambda} \to \begin{bmatrix} 1 & 0 & c/2 & b \\ 0 & 1 & b/2 & a \end{bmatrix} \to \begin{bmatrix} 1 & 0 & c/2 & b \\ 0 & 2 & b & 2a \end{bmatrix} = \Lambda^*$$

The dual to Prym$^*(C,\sigma)$ is again Prym(C,σ).

Theorem

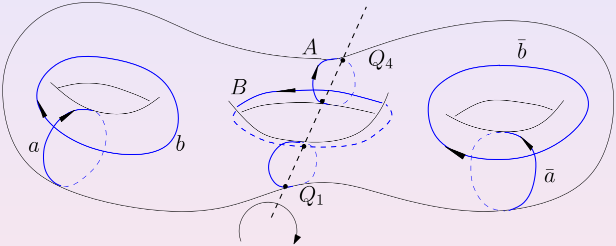
$\exists$ birationally non-equivalent genus 2 curves $\tilde\Gamma_1, \tilde\Gamma_2, \tilde\Gamma_3$ and $\Gamma_1, \Gamma_2, \Gamma_3$ such that the following coverings hold

$$
\begin{array}{ccccc}
\mathrm{Jac}(\widetilde\Gamma_1) & & \mathrm{Jac}(\widetilde\Gamma_2) & & \mathrm{Jac}(\widetilde\Gamma_3) \\
\| & \overset{2:1}{\searrow} & \downarrow & \overset{2:1}{\swarrow} & \| \\
{\scriptstyle 4:1}\ \| & & \mathrm{Prym}(C,\sigma) & & \|\ {\scriptstyle 4:1} \\
\Downarrow & \overset{2:1}{\swarrow} & \downarrow {\scriptstyle 2:1} & \overset{2:1}{\searrow} & \Downarrow \\
\mathrm{Jac}(\Gamma_1) & & \mathrm{Jac}(\Gamma_2) & & \mathrm{Jac}(\Gamma_3) \\
\| & \overset{2:1}{\searrow} & \downarrow & \overset{2:1}{\swarrow} & \| \\
{\scriptstyle 4:1}\ \| & & \mathrm{Prym}^*(C,\sigma) & & \|\ {\scriptstyle 4:1} \\
\Downarrow & \swarrow & \downarrow & \overset{2:1}{\searrow} & \Downarrow \\
\mathrm{Jac}(\widetilde\Gamma_1) & & \mathrm{Jac}(\widetilde\Gamma_2) & & \mathrm{Jac}(\widetilde\Gamma_3)
\end{array}
$$

The diagram of *Richelot correspondences* (arrows denote the duplication $(I\ \tau) \to (I\ 2\tau)$):

$$
\begin{array}{ccccc}
\mathrm{Jac}(\widetilde\Gamma_1) & & \mathrm{Jac}(\widetilde\Gamma_2) & & \mathrm{Jac}(\widetilde\Gamma_3) \\
\uparrow & \nwarrow & & \searrow & \downarrow \\
\mathrm{Jac}(\Gamma_1) & & \mathrm{Jac}(\Gamma_2) & & \mathrm{Jac}(\Gamma_3)
\end{array}
$$

The involution $\sigma : C \to C$ with 4 fixed points $Q_1, \dots, Q_4$.
Then C is a 2-fold covering $\pi : C \to E$, $E = C/\sigma$ (elliptic curve), ramified at $Q_1, \dots, Q_4$.



Action on cicles: $\sigma(a) = -\bar{a}$, $\sigma(A) = -A$, $\sigma(b) = -\bar{b}$, $\sigma(B) = -B$,

Let $u, w, \bar{u}$ be the corresponding *normalized* holomorphic differentials, such that $\sigma^*(u) = -\bar{u}$, $\sigma^*(\bar{u}) = -u$, $\sigma^*(w) = -w$ and

$$\begin{bmatrix} \oint_a u & \oint_A u & \oint_{\bar{a}} u \\ \oint_a w & \oint_A w & \oint_{\bar{a}} w \\ \oint_a \bar{u} & \oint_A \bar{u} & \oint_{\bar{a}} \bar{u} \end{bmatrix} = \mathbf{I}, \quad \mathcal{B} = \begin{bmatrix} \oint_b u & \oint_B u & \oint_{\bar{b}} u \\ \oint_b w & \oint_B w & \oint_{\bar{b}} w \\ \oint_b \bar{u} & \oint_B \bar{u} & \oint_{\bar{b}} \bar{u} \end{bmatrix}.$$

Extracting the Prym variety from Jac(C) (II)

Apply 2 transformations of degree 2 on the period matrix:

$$(\mathbb{I}\,\mathcal{B}) = (V_1 \cdots V_6) \longrightarrow \left\{ \begin{array}{l} V_4' = V_4 + V_6 \\ V_6' = V_4 - V_6 \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} V_1' = V_1 + V_3 \\ V_3' = V_1 - V_3 \end{array} \right\}$$

$$\longrightarrow \begin{bmatrix} 2 & 0 & 0 & 2\pi & 2p & 0 \\ 0 & 1 & 0 & 2p & 2P & 0 \\ 0 & 0 & 1 & 0 & 0 & \tau \end{bmatrix} \leftarrow \text{complete splitting}$$

Hence Jac(C) contains the (2,1) polarized Abelian subvariety Prym(C/σ) with the normalized period matrix

$$\Lambda = \begin{bmatrix} 2 & 0 & \Pi \\ 0 & 1 & \end{bmatrix}, \quad \Pi = \begin{pmatrix} 2\pi & 2p \\ 2p & 2P \end{pmatrix}$$

The puzzle of six Jacobians

Consider two genus 3 curves

$$
\begin{aligned}
C : y^2 &= P_2(x) + 2h_3\sqrt{\Phi(x)}, & \Phi(x) &= (x-c_1)(x-c_2)(x-c_3)(x-c_4), \\
K : y^2 &= P_2(x) + \sqrt{\Psi(x)}, & \Psi(x) &= P_2^2(x) - 4h_3^2\Phi(x) \\
& & &= k(x-s_1)(x-s_2)(x-s_3)(x-s_4),
\end{aligned}
$$

The coverings of even-order elliptic curves:

$$
\begin{aligned}
C &\longrightarrow E = \{z^2 = \Phi(x)\} \quad \text{ramified at } \; Q_i = (s_i, \sqrt{\Phi(s_i)}), \\
K &\longrightarrow \mathcal{E} = \{z^2 = \Psi(x)\} \quad \text{ramified at } \; \mathcal{P}_i = (c_i, \sqrt{\Psi(c_i)})
\end{aligned}
\qquad \text{i=1,2,3,4}
$$

L. Heine: Prym(C,σ) and Prym(K,σ) are *dual*.

$$
\begin{array}{ccccc}
\mathrm{Jac}(\widetilde{\Gamma}_1) & & \mathrm{Jac}(\widetilde{\Gamma}_2) & & \mathrm{Jac}(\widetilde{\Gamma}_3) \\
\| & \searrow & \downarrow & \swarrow & \| \\
\| & & \mathrm{Prym}(C,\sigma) & & \| \\
\Downarrow & \swarrow & \downarrow & \searrow & \Downarrow \\
\mathrm{Jac}(\Gamma_1) & & \mathrm{Jac}(\Gamma_2) & & \mathrm{Jac}(\Gamma_3) \\
 & \searrow & \downarrow & \swarrow & \\
 & & \mathrm{Prym}(K,\sigma) & & \\
 & \swarrow & \downarrow & \searrow & \\
\mathrm{Jac}(\widetilde{\Gamma}_1) & & \mathrm{Jac}(\widetilde{\Gamma}_2) & & \mathrm{Jac}(\widetilde{\Gamma}_3)
\end{array}
$$

The puzzle of six Jacobians (II)

The 3 curves $\widetilde{\Gamma} = \{ w^2 = z(z - d_1^2)(z - d_2^2)(z - d_3^2)(z - d_1^2 d_2^2 d_3^2) \}$

$$d_\alpha^2 = \left(\frac{\sqrt{S_{14}^\alpha} + \sqrt{S_{24}^\alpha}}{\sqrt{S_{12}^\alpha}} \right)^2 \quad \alpha = 1, 2, 3,$$

$$\text{or} \quad d_\alpha^2 = \left(\frac{\sqrt{S_{12}^\alpha} + \sqrt{S_{24}^\alpha}}{\sqrt{S_{14}^\alpha}} \right)^2 \text{ or } \quad d_\alpha^2 = \left(\frac{\sqrt{S_{12}^\alpha} + \sqrt{S_{14}^\alpha}}{\sqrt{S_{24}^\alpha}} \right)^2,$$

$$S_{12}^\alpha = (s_1 - s_2)(s_3 - s_4) \cdot [(s_2 - s_4)^2 (s_3 - c_\alpha)(s_3 - c_4)(s_1 - c_\alpha)(s_1 - c_4) + (s_1 - s_3)^2 (s_2 - c_\alpha)(s_2 - c_4)(s_4 - c_\alpha)(s_4 - c_4)],$$

$S_{14}^\alpha, S_{24}^\alpha$ obtained from S_{12}^α by permutations of $\{s_1, s_2, s_3, s_4\}$.

The 3 curves $\Gamma = \{ w^2 = z(z - k_1^2)(z - k_2^2)(z - k_3^2)(z - k_1^2 k_2^2 k_3^2) \}$
The expressions for k_1, k_2, k_3 are obtained from those of d_1, d_2, d_3 by the permutation $\{s_1, s_2, s_3, s_4\} \leftrightarrow \{c_1, c_2, c_3, c_4\}$.

Example: the Clebsch integrable case of the Kirchoff equations on $e(3) = \{K, p\} = \mathbb{R}^6$

$$\dot{K} = K \times \frac{\partial H}{\partial K} + p \times \frac{\partial H}{\partial p},$$
$$\dot{p} = p \times \frac{\partial H}{\partial K}$$

with the Clebsch Hamiltonian

$$H_1 = \frac{1}{2}(c_1 K_1^2 + c_2 K_2^2 + c_3 K_3^2) + \frac{1}{2}(b_1 p_1^2 + b_2 p_2^2 + b_3 p_3^2),$$
$$\frac{b_1 - b_2}{c_3} + \frac{b_2 - b_3}{c_1} + \frac{b_3 - b_1}{c_2} = 0$$

and with the 3×3 Lax representation with an *elliptic* spectral parameter

$$\dot{L}(r) = [L(r), A(r)], \quad L_{ij}(r) = \varepsilon_{ijk}\left(\sqrt{r - c_k}\, K_k + \sqrt{(r - c_i)(r - c_j)}\, p_k\right).$$

The spectral curve $|L(r) - \mathbf{I}y| = 0$:

$$C : \{y^2 = h_2 r^2 + h_1 r + h_0 + 2h_3\sqrt{(r - c_1)(r - c_2)(r - c_3)}\}.$$

The spectral curve

$$C : \left\{ y^2 = h_2 r^2 + h_1 r + h_0 + 2h_3 \sqrt{\underbrace{(r-c_1)(r-c_2)(r-c_3)}_{\Psi(r)}} \right\}$$

covers the elliptic curve $E : \{w^2 = (r-c_1)(r-c_2)(r-c_3)\}$.
The 2-fold covering $C \to E$ is ramified at 4 points $Q_i = (s_i, \sqrt{\Psi(s_i)})$, s_i being the roots of the polynomial

$$\psi(r) = (h_2 r^2 + h_1 r + h_0)^2 - 4h_3^2 \Psi(r) = (r-s_1)(r-s_2)(r-s_3)(r-s_4)\,.$$

Hence genus(C)=3 (The Riemann–Hurwitz formula)
C admits an involution $\sigma : C \to C$ with 4 fixed points $Q_1, \dots, Q_4$.

• L. Heine (1983): The complex (2-dim.) generic invariant varieties of the Clebsch system are open subsets of the Prym varieties Prym(C/σ) $\subset$ Jac(C) (the anti-symmetric part of Jac(C) under σ).

• On the other hand, F. Kötter (1891) integrated the Clebsch system by using the genus 2 hyperelliptic functions (no Pryms !).

Ueber die Bewegung eines festen Körpers in einer Flüssigkeit.

(Von Herrn *Fritz Kötter*.)

α denselben Werth beilegen und dann die Wurzeln $\sqrt{s_\beta - c_\alpha}$ entsprechend der eben angegebenen Bedingung auswählen. Ferner soll gesetzt werden:

$$(24.)\quad \begin{cases} \xi_\alpha = x_\alpha\left(\dfrac{\sqrt{(s_1-c_1)(s_1-c_2)(s_1-c_3)}}{\sqrt{s_1-c_\alpha}\sqrt{\psi'(s_1)}} + i\dfrac{\sqrt{(s_2-c_1)(s_2-c_2)(s_2-c_3)}}{\sqrt{s_2-c_\alpha}\sqrt{\psi'(s_2)}}\right) \\ \qquad + y_\alpha\left(\dfrac{\sqrt{s_1-c_\alpha}}{\sqrt{\psi'(s_1)}} + i\dfrac{\sqrt{s_2-c_\alpha}}{\sqrt{\psi'(s_2)}}\right), \\ \eta_\alpha = x_\alpha\left(\dfrac{\sqrt{(s_1-c_1)(s_1-c_2)(s_1-c_3)}}{\sqrt{s_1-c_\alpha}\sqrt{\psi'(s_1)}} - i\dfrac{\sqrt{(s_2-c_1)(s_2-c_2)(s_2-c_3)}}{\sqrt{s_2-c_\alpha}\sqrt{\psi'(s_2)}}\right) \\ \qquad + y_\alpha\left(\dfrac{\sqrt{s_1-c_\alpha}}{\sqrt{\psi'(s_1)}} - i\dfrac{\sqrt{s_2-c_\alpha}}{\sqrt{\psi'(s_2)}}\right), \end{cases}$$

$$(25.)\quad d_\alpha = \frac{\dfrac{\sqrt{s_3-c_\alpha}}{\sqrt{\psi'(s_3)}} + i\dfrac{\sqrt{s_4-c_\alpha}}{\sqrt{\psi'(s_4)}}}{\dfrac{\sqrt{s_1-c_\alpha}}{\sqrt{\psi'(s_1)}} + i\dfrac{\sqrt{s_2-c_\alpha}}{\sqrt{\psi'(s_2)}}}, \quad -d_\alpha^{-1} = \frac{\dfrac{\sqrt{s_3-c_\alpha}}{\sqrt{\psi'(s_3)}} - i\dfrac{\sqrt{s_4-c_\alpha}}{\sqrt{\psi'(s_4)}}}{\dfrac{\sqrt{s_1-c_\alpha}}{\sqrt{\psi'(s_1)}} - i\dfrac{\sqrt{s_2-c_\alpha}}{\sqrt{\psi'(s_2)}}}.$$

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$$(30.)\quad Z_\beta = \sqrt{z_\beta(z_\beta-d_1^2)(z_\beta-d_2^2)(z_\beta-d_3^2)\left(\frac{z_\beta}{d_1^2 d_2^2 d_3^2}-1\right)} \quad (\beta = 1, 2),$$